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A New Approach to Numerical Computation of Hausdorff Dimension of Iterated Function Systems: Applications to Complex Continued Fractions

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Abstract. In a previous paper (Falk and Nussbaum, in C^m Eigenfunctions of Perron–Frobenius operators and a new approach to numerical computation of hausdorff dimension: applications in $\mathbb{R}^1$, 2016. ArXiv e-prints arXiv:1612.00870), the authors developed a new approach to the computation of the Hausdorff dimension of the invariant set of an iterated function system or IFS and studied some applications in one dimension. The key idea, which has been known in varying degrees of generality for many years, is to associate to the IFS a parametrized family of positive, linear, Perron-Frobenius operators L_s. In our context, L_s is studied in a space of C^m functions and is not compact. Nevertheless, it has a strictly positive C^m eigenfunction v_s with positive eigenvalue λ_s equal to the spectral radius of L_s. Under appropriate assumptions on the IFS, the Hausdorff dimension of the invariant set of the IFS is the value $s = s_*$ for which $\lambda_s = 1$. To compute the Hausdorff dimension of an invariant set for an IFS associated to complex continued fractions, (which may arise from an infinite iterated function system), we approximate the eigenvalue problem by a collocation method using continuous piecewise bilinear functions. Using the theory of positive linear operators and explicit a priori bounds on the partial derivatives of the strictly positive eigenfunction v_s, we are able to give rigorous upper and lower bounds for the Hausdorff dimension s_*, and these bounds converge to s_* as the mesh size approaches zero. We also demonstrate by numerical computations that improved estimates can be obtained by the use of higher order piecewise tensor product polynomial approximations, although the present theory does not guarantee that these are strict upper and lower bounds. An important feature of our approach is that it also applies to the much more general problem of computing approximations to the spectral radius of positive transfer operators, which arise in many other applications.

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1. Introduction

Our interest in this paper is in describing methods which give rigorous estimates for the Hausdorff dimension of invariant sets for (possibly infinite) iterated function systems or IFS's. For simplicity, we do not consider here the important case of graph directed iterated function systems, for which a similar approach can be given. Our immediate application is to the case of invariant sets for IFS's associated to complex continued fractions, but we expect to show in future work that other interesting examples can also be treated. In previous work [13], we considered IFS's in one dimension, and in particular the computation of the Hausdorff dimension of some Cantor sets arising from continued fraction expansions and also other examples in which the underlying maps have less regularity.

To describe our present results, we first recall some general facts about iterated function systems. Let D be a complete metric space with metric ρ, and $\theta_b : D \to D$, $b \in \mathcal{B}$, a contraction mapping, i.e., a Lipschitz mapping (with respect to ρ) with Lipschitz constant $\mathrm{Lip}(\theta_b)$, satisfying $\mathrm{Lip}(\theta_b) := c_b < 1$. If $\mathcal{B}$ is finite and the above assumption holds, it is known (see Section 3 of [22] or Chapter 2, Section 2 of [11]) that there exists a unique, compact, nonempty set $C \subset D$ such that $C = \cup_{b\in\mathcal{B}}\theta_b(C)$. The set C is called the invariant set for the IFS $\{\theta_b : b \in \mathcal{B}\}$. If $\mathcal{B}$ is infinite and $\sup\{c_b : b \in \mathcal{B}\} = c < 1$, there is a naturally defined nonempty invariant set $C \subset D$ such that $C = \cup_{b\in\mathcal{B}}\theta_b(C)$, but C need not be compact (e.g., see [36] or [37]). In [13], the index set $\mathcal{B}$ was finite and could be simply described by the notation θ_j, $j = 1,\ldots,m$. In the case of complex continued fractions, which we consider here, $b = m + ni$, m belonging to a subset of $\mathbb{N}$ and n belonging to a subset of $\mathbb{Z}$.

Although we shall eventually specialize, since the method we consider has applications other than the one we describe in this paper, it is useful, as was done in [13], to describe initially some function analytic results in the generality of the previous paragraph. Let H be a bounded, open, mildly regular (defined in Sect. 4) subset of $\mathbb{R}^n$ and let $C^k_{\mathbb{C}}(\bar{H})$ denote the complex Banach space of C^k complex-valued maps, all of whose partial derivatives of order $\nu \le k$ extend continuously to $\bar{H}$. For a given positive integer N, assume that $g_b : \bar{H} \to (0,\infty)$ are strictly positive C^N functions for $b \in \mathcal{B}$ and $\theta_b : \bar{H} \to \bar{H}$, $b \in \mathcal{B}$, are C^N maps and contractions. For $s > 0$ and integers k, $0 \le k \le N$, one can define a bounded linear map $L_{s,k} : C^k(\bar{H}) \to C^k(\bar{H})$ by the formula

$$(L_{s,k} f)(x) = \sum_{b\in\mathcal{B}} [g_b(x)]^s f(\theta_b(x)). \tag{1.1}$$

Note that (1.1) also defines a bounded linear map of $C^k_{\mathbb{R}}(\bar{H})$ to itself, which (abusing notation), we shall also denote by $L_{s,k}$. Linear maps like $L_{s,k}$ are sometimes called positive transfer operators or Perron-Frobenius operators and arise in many contexts other than computation of Hausdorff dimension: see, for example, [1]. If $r(L_{s,k})$ denotes the spectral radius of $L_{s,k}$, then $\lambda_s = r(L_{s,k})$ is positive and independent of k for $0 \le k \le N$; and λ_s is an algebraically simple eigenvalue of $L_{s,k}$ with a corresponding unique, normalized strictly positive eigenfunction $v_s \in C^N(\bar{H})$. Furthermore, the map $s \mapsto \lambda_s$ is continuous. If $\sigma(L_{s,k}) \subset \mathbb{C}$ denotes the spectrum of $L_{s,k}$, $\sigma(L_{s,k})$ depends on k, but for $1 \le k \le N$,

$$\sup\{|z| : z \in \sigma(L_{s,k}) \setminus \{\lambda_s\}\} < \lambda_s. \tag{1.2}$$

If $k = 0$, the strict inequality in (1.2) may fail. A more general version of the above result is stated in Theorem 4.1 of this paper and Theorem 4.1 is a special case of results in [43]. The method of proof involves ideas from the theory of positive linear operators, particularly generalizations of the Kreĭn-Rutman theorem to noncompact linear operators; see [2,31,35,40,41,43,49]. Although the example of complex continued fractions that we study in this paper leads to an analytic IFS, we also have in mind allowing perturbations to a C^m IFS (e.g., as done in Section 5 of [13]). Hence, we work in a Banach space of C^m functions. Note however, that the particular problem in this paper can also be set up in a Banach space of analytic functions in two complex variables (see [37]).

The linear operators which are relevant for the computation of Hausdorff dimension comprise a small subset of the transfer operators described in (1.1), but the analysis problem which we shall consider here can be described in the generality of (1.1) and is of interest in this more general context. We want to find rigorous methods to estimate $r(L_{s,k})$ accurately and then use these methods to estimate s_*, where, in our applications, s_* will be the unique number $s \ge 0$ such that $r(L_{s,k}) = 1$. Under further assumptions, we shall see that s_* equals $\dim_H(C)$, the Hausdorff dimension of the invariant set associated to the IFS. This observation about Hausdorff dimension has been made, in varying degrees of generality by many authors. See, for example, [4–6,9–11,17,19–24,26,36,37,44,46–48,50].

We assume in this paper that H is a bounded, open mildly regular subset of $\mathbb{R}^2 = \mathbb{C}$ and that θ_b, $b \in \mathcal{B}$, are analytic or conjugate analytic contraction maps, defined on an open neighborhood of $\bar{H}$ and satisfying $\theta_b(H) \subset H$. We define $D\theta_b(z) = \lim_{h \to 0} |[\theta_b(z+h) - \theta_b(z)]/h|$, where $h \in \mathbb{C}$ in the limit, and we assume that $D\theta_b(z) \neq 0$ for $z \in \bar{H}$. In this case, $L_{s,k}$ is defined by (1.1), with x replaced by z, and $g_b(z) = D\theta_b(z)$. It is then possible to obtain explicit upper and lower bounds for $D_1^p v_s(x_1,x_2))/v_s(x_1,x_2)$ and $D_2^p v_s(x_1,x_2))/v_s(x_1,x_2)$, where $D_1 = \partial/\partial x_1$ and $D_2 = \partial/\partial x_2$. However, for simplicity we restrict ourselves to the choice $\theta_b(z) = (z+b)^{-1}$, where $b \in \mathbb{C}$ and $\mathrm{Re}(b) > 0$. In this case we obtain in Sect. 5 explicit upper and lower bounds for $D_k^p v_s(x_1,x_2))/v_s(x_1,x_2)$ for $1 \le p \le 4$, $1 \le k \le 2$, and $x_1 > 0$. In both the one and two dimensional cases, these estimates play a crucial role in allowing us to obtain rigorous upper and lower bounds for the Hausdorff dimension. Of course, obtaining these estimates adds to the length of [13] and this paper. However, aside from their intrinsic interest, we believe these results will prove useful in other contexts, e.g., in treating generalizations of the *Texan conjecture* (see [23,28]).

The basic idea of our numerical scheme is to cover $\bar{H}$ by nonoverlapping squares of side h. We remark that our collection of squares need not be a *Markov partition* for our IFS; compare [38]. We then approximate the strictly positive, C^2 eigenfunction v_s by a continuous piecewise bilinear function. Using the explicit bounds on the unmixed derivatives of v_s of order 2, we are then able to associate to the operator $L_{s,k}$, square matrices A_s and B_s, which have nonnegative entries and also have the property that $r(A_s) \leq \lambda_s \leq r(B_s)$. A key role here is played by an elementary fact (see Lemma 2.2 in Sect. 2) which is not as well known as it should be and in the matrix case reduces to the following observation: If M is a nonnegative matrix and v is a strictly positive vector and $Mv \leq \lambda v$, (coordinate-wise), then $r(M) \leq \lambda$. Analogously, $r(M) \geq \lambda$ if $Mv \geq \lambda v$.

If s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, so that s_* is the Hausdorff dimension of the invariant set for the IFS under study, we proceed as follows. If we can find a number s_1 such that $r(B_{s_1}) \leq 1$, then, since the map $s \mapsto \lambda_s$ is decreasing, $\lambda_{s_1} \leq r(B_{s_1}) \leq 1$, and we can conclude that $s_* \leq s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \geq 1$, then $\lambda_{s_2} \geq r(A_{s_2}) \geq 1$, and we can conclude that $s_* \geq s_2$. By choosing the mesh size for our approximating piecewise polynomials to be sufficiently small, we can make $s_1 - s_2$ small, providing a good estimate for s_*. For a given s, $r(A_s)$ and $r(B_s)$ are easily found by variants of the power method for eigenvalues, since the largest eigenvalue of A_s (respectively, of B_s) has multiplicity one and is the only eigenvalue of its modulus. When the IFS is infinite, the procedure is somewhat more complicated, and we include the necessary theory to deal with this case.

This new approach was illustrated in [13] by first considering the computation of the Hausdorff dimension of invariant sets in $[0, 1]$ arising from classical continued fraction expansions. In this much studied case, one defines $\theta_m(x) = 1/(x + m)$, for m a positive integer and $x \in [0, 1]$; and for a subset $\mathcal{B} \subset \mathbb{N}$, one considers the IFS $\{\theta_m : m \in \mathcal{B}\}$ and seeks estimates on the Hausdorff dimension of the invariant set $C = C(\mathcal{B})$ for this IFS. This problem has previously been considered by many authors. See [3,5,6,17–21,23,24]. In this case, (1.1) becomes

$$(L_{s,k}v)(x) = \sum_{m\in\mathcal{B}} \left(\frac{1}{x+m}\right)^{2s} v\left(\frac{1}{x+m}\right), \qquad 0 \leq x \leq 1,$$

and one seeks a value $s \geq 0$ for which $\lambda_s := r(L_{s,k}) = 1$.

In Sect. 3, we consider the computation of the Hausdorff dimension of some invariant sets arising from complex continued fractions. Suppose that $\mathcal{B}$ is a subset of $I_1 := \{m + ni : m \in \mathbb{N}, n \in \mathbb{Z}\}$, and for each $b \in \mathcal{B}$, define $\theta_b(z) = (z + b)^{-1}$. Note that θ_b maps $\bar{G} = \{z \in \mathbb{C} : |z - 1/2| \leq 1/2\}$ into itself. We are interested in the Hausdorff dimension of the invariant set $C = C(\mathcal{B})$ for the IFS $\{\theta_b : b \in \mathcal{B}\}$. This is a two dimensional problem and we allow the possibility that $\mathcal{B}$ is infinite. In general (contrast work in [24] and [23]), it does not seem possible in this case to replace $L_{s,k}$, $k \geq 2$, by an operator Λ_s acting on a Banach space of analytic functions of one complex variable and satisfying $r(\Lambda_s) = r(L_{s,k})$. We note that it is possible to set this problem up in a Banach space of two complex variables (c.f. [37]). Instead, we work in $C^2(\bar{G})$ and apply our methods to obtain rigorous upper and lower bounds for the Hausdorff dimension $\dim_H(C(\mathcal{B}))$ for several examples. The case $\mathcal{B} = I_1$ has been of particular interest and is one motivation for this paper. In [16], Gardner and Mauldin proved that $d := \dim_H(C(I_1)) < 2$. In Theorem 6.6 of [36], Mauldin and Urbański proved that $1.2484 \leq d \leq 1.885$, and in [45], Priyadarshi proved that $d \geq 1.78$. In Sect. 3.2, we show (modulo roundoff errors in the calculation) that $1.85574 \leq d \leq 1.85589$. We believe (see Remark 3.1 in Sect. 3) that this estimate can be made rigorous by using interval arithmetic along with high order precision, although since we consider this paper to be a feasibility study, we have not done this.

In the case when the eigenfunctions v_s have additional smoothness, it is natural to approximate $v_s(\cdot)$ by piecewise tensor product polynomials of higher degree. In this situation, the corresponding matrices A_s and B_s may no longer have all nonnegative entries and so the arguments of this paper are no longer directly applicable. However, as demonstrated in Table 2 and Table 3, this approach gives much improved estimates for the value of s for which $r(L_s) = 1$. It is our intent to develop an extension of our theory to make these into rigorous bounds.

It is also worth comparing the approach used in our paper with that of McMullen [38]. Superficially the methods seem different, but there are underlying connections. We exploit the existence of a C^k, strictly positive eigenfunction v_s of (1.1) with eigenvalue λ_s equal to the spectral radius of $L_{s,k}$; and we observe that explicit bounds on derivatives of v_s can be exploited to prove convergence rates on numerical approximation schemes which approximate λ_s. McMullen does not explicitly mention the operator $L_{s,k}$ or the analogue of $L_{s,k}$ for graph directed iterated function systems, and he does not use C^k, strictly positive eigenfunctions of equations like (1.1) or obtain bounds on partial derivatives of such positive eigenfunctions. Instead, he exploits finite positive measures μ which are called "$\mathcal{F}-$invariant densities of dimension δ." If s_* is a value of s for which the above eigenvalue $\lambda_s = 1$, then in our context the measure μ is an eigenfunction of the Banach space adjoint $(L_{s_*,0})^*$ with eigenvalue 1, and our s_* corresponds to δ above. Standard arguments using weak* compactness, the Schauder-Tychonoff fixed point theorem, and the Riesz representation theorem imply the existence of a regular, finite, positive, complete measure μ, defined on a σ-algebra containing all Borel subsets of the underlying space $\bar{H}$ and such that $(L_{s_*,0})^*\mu = \mu$ and $\int v_{s_*}\, d\mu = 1$.

McMullen also uses refinements of *Markov partitions,* while our partitions, both here and in [13], need not be Markov. However, in the end, both approaches generate (different) $n \times n$ nonnegative matrices M_s, parametrized by a parameter s and both methods use the spectral radius of M_s to approximate the desired Hausdorff dimension s_*. McMullen's matrices are obtained by approximating certain nonconstant functions defined on a refinement of the original Markov partition by piecewise constant functions defined with respect to this refinement. We approximate by bilinear functions on each subset in our partition. As we show below, by exploiting estimates on higher derivatives of $v_s(\cdot)$, our methods give explicit upper and lower bounds for s_* and more rapid convergence to s_* than one obtains using piecewise constant approximations.

The square matrices A_s and B_s mentioned above and described in more detail later in the paper have nonnegative entries and satisfy $r(A_s) \le \lambda_s \le r(B_s)$. To apply standard numerical methods, it is useful to know that all eigenvalues $\mu \neq r(A_s)$ of A_s satisfy $|\mu| < r(A_s)$ and that $r(A_s)$ has algebraic multiplicity one and that corresponding results hold for $r(B_s)$. Such results were proved in Section 7 of [13] in the one dimensional case when the mesh size, h, is sufficiently small, and a similar argument can be used in the two dimensional case under study here. Note that this result does not follow from the standard theory of nonnegative matrices, since A_s and B_s typically have zero columns and are not primitive. As in [13], we can also prove that $r(A_s) \le r(B_s) \le (1 + C_1 h^2) r(A_s)$, where the constant C_1 can be explicitly estimated. Once it is known that L_s has a strictly positive eigenfunction, v_s, with eigenvalue $\lambda_s := r(L_s)$, the log convexity of $s \mapsto \lambda_s$ and the fact that the map is strictly decreasing follow easily by an argument given in [42]. See, also [37]. This same result holds for $s \mapsto r(M_s)$, where M_s is a naturally defined matrix such that $A_s \le M_s \le B_s$. This idea is exploited in our computer code in the following way. Recall that if we can find a number s_1 such that $r(B_{s_1}) \le 1$, then, since the map $s \mapsto \lambda_s$ is decreasing, $\lambda_{s_1} \le r(B_{s_1}) \le 1$, and we can conclude that $s_* \le s_1$. To obtain the best bound, we seek a value s_1 such that $r(B_{s_1})$ is as close as possible to 1, while still remaining ≤ 1. This is done by a slight modification of the secant method applied to finding a zero of the function $\log[r(B_{s_1})]$. A similar approach is used with A_s to find a lower bound for s_*.

Since the posting of our work on the arXiv [12], several authors have taken up the issue of obtaining rigorous upper and lower bounds on the Hausdorff dimension. In [25], Jenkinson and Pollicott modified methods from their 2001 paper [24] to rigorously compute the Hausdorff dimension of E[1,2] to 100 decimal places. In [7], the authors employ the computational approach developed in [12] and [13] to obtain rigorous estimates for the Hausdorff dimension of continued fractions whose entries are restricted to infinite sets.

A summary of the paper is as follows. In Sect. 2, we recall the definition of Hausdorff dimension and present some mathematical preliminaries. In Sect. 3, we present the details of our approximation scheme for Hausdorff dimension, explain the crucial role played by estimates on unmixed partial derivatives of order ≤ 2 of v_s, and give the aforementioned estimates for Hausdorff dimension. We emphasize that this is a feasibility study. We have limited the accuracy of our approximations to what is easily found using the standard precision of *Matlab* and have run only a limited number of examples, using mesh sizes that allow the programs to run fairly quickly. In addition, we have not attempted to exploit the special features of our problems, such as the fact that our matrices are sparse. Thus, it is clear that one could write a more efficient code that would also speed up the computations. However, the *Matlab* programs we have developed are available on the web at www.math.rutgers.edu/~falk/hausdorff/codes.html, and we hope other researchers will run other examples of interest to them.

The theory underlying the work in Sect. 3 is presented in Sects. 4–7. In Sect. 4 we describe some results concerning existence of C^m positive eigenfunctions for a class of positive (in the sense of order-preserving) linear operators. We remark that Theorem 4.1 in Sect. 4 was only proved in [43] for finite IFS's. As a result, some care is needed in dealing with infinite IFS's. In Sect. 5, we derive explicit bounds on the partial derivatives of eigenfunctions of operators in which the mappings θ_b are given by Möbius transformations which map a given bounded open subset H of $\mathbb{C} := \mathbb{R}^2$ into H. We use this information in Theorems 5.10–5.13 to obtain results about the case of infinite IFS's which are adequate for our immediate purposes. In Sect. 6, we verify some spectral properties of the approximating matrices which justify standard numerical algorithms for computing their spectral radii. Finally, in Sect. 7, we discuss the log convexity of the spectral radius $r(L_s)$, which we exploit in our numerical approximation scheme.

2. Preliminaries

We recall the definition of the Hausdorff dimension, $\dim_H(K)$, of a subset $K \subset \mathbb{R}^N$. For a given $s \geq 0$ and each set $K \subset \mathbb{R}^N$, one defines

$$H^s_\delta(K) = \inf \left\{ \sum_i |U_i|^s : \{U_i\} \text{ is a } \delta \text{ cover of } K \right\},$$

where $|U|$ denotes the diameter of U and a countable collection $\{U_i\}$ of subsets of $\mathbb{R}^N$ is a δ-cover of $K \subset \mathbb{R}^N$ if $K \subset \cup_i U_i$ and $0 < |U_i| < \delta$ for all i. One then defines the s-dimensional Hausdorff measure

$$H^s(K) = \lim_{\delta \to 0+} H^s_\delta(K).$$

Finally, the Hausdorff dimension of K, $\dim_H(K)$, is defined as

$$\dim_H(K) = \inf\{s : H^s(K) = 0\}.$$

We now state the main result connecting Hausdorff dimension to the spectral radius of the map defined by (1.1). To do so, we first define the concept of an *infinitesimal similitude*. Let (S, d) be a bounded, complete, perfect metric space. If $\theta : S \to S$, then θ is an infinitesimal similitude at $t \in S$ if for any sequences $(s_k)_k$ and $(t_k)_k$ with $s_k \neq t_k$ for $k \geq 1$ and $s_k \to t$, $t_k \to t$, the limit

$$\lim_{k \to \infty} \frac{d(\theta(s_k), \theta(t_k))}{d(s_k, t_k)} =: (D\theta)(t)$$

exists and is independent of the particular sequences $(s_k)_k$ and $(t_k)_k$. Furthermore, θ is an infinitesimal similitude on S if θ is an infinitesimal similitude at t for all $t \in S$.

This concept generalizes the concept of affine linear similitudes, which are affine linear contraction maps θ satisfying for all $x, y \in \mathbb{R}^n$

$$d(\theta(x), \theta(y)) = cd(x, y), \quad c < 1.$$

In particular, the examples discussed in [13], such as maps of the form $\theta(x) = 1/(x + m)$, with m a positive integer, are infinitesimal similitudes. More generally, if S is a compact subset of $\mathbb{R}^1$ and $\theta : S \to S$ extends to a C^1 map defined on an open neighborhood of S in $\mathbb{R}^1$, then θ is an infinitesimal similitude. If S is a compact subset of $\mathbb{R}^2 := \mathbb{C}$ and $\theta : S \to S$ extends to an analytic or conjugate analytic map defined on an open neighborhood of S in $\mathbb{C}$, θ is an infinitesimal similitude.

Theorem 2.1. (Theorem 1.2 of [44]) *Let $\theta_i : S \to S$ for $1 \le i \le N$ be infinitesimal similitudes and assume that the map $t \mapsto (D\theta_i)(t)$ is a strictly positive Hölder continuous function on S. Assume that θ_i is a Lipschitz map with Lipschitz constant $c_i \le c < 1$ and let C denote the unique, compact, nonempty invariant set such that*

$$C = \cup_{i=1}^N \theta_i(C).$$

Further, assume that θ_i satisfy

$$\theta_i(C) \cap \theta_j(C) = \emptyset, \text{ for } 1 \le i, j \le N.\ i \ne j$$

and are one-to-one on C. Then the Hausdorff dimension of C is given by the unique σ_0 such that $r(L_{\sigma_0}) = 1$, where $L_s : C(S) \to C(S)$ is defined for $s \ge 0$ by

$$(L_s f)(t) = \sum_{i=1}^N [D\theta_i(t)]^s f(\theta_i(t)).$$

Furthermore, L_s has a strictly positive Hölder continuous eigenfunction with eigenvalue equal to the spectral radius of L_s.

A proof of the existence of the set C in this generality can be found in [11] generalizing earlier work of [22]. The remainder of the theorem, aside from the eigenfunction, can be derived from the work of Rugh [48]. Other related results can be found in [4,11,22,37,47,50].

The following lemma is a well-known result, but we sketch the proof because the lemma with play a crucial role in some of our later arguments.

Lemma 2.2. *Let Q be a compact Hausdorff space, $X = C_{\mathbb{R}}(Q)$, the Banach space of continuous, real-valued functions $f : Q \to \mathbb{R}$ in the* sup *norm,*

$$K = \{f \in X : f(t) \ge 0\ \forall t \in Q\}, \text{ and } \operatorname{int}(K) = \{f \in X : f(t) > 0\ \forall t \in Q\}.$$

If $f, g \in X$, write $f \le g$ if $g - f \in K$. Let $L : X \to X$ be a bounded linear map such that $L(K) \subset K$ and write $r(L) := \lim_{n\to\infty} \|L^n\|^{1/n}$, the spectral radius of L. If there exists $w \in \operatorname{int}(K)$ such that $Lw \le \beta w$ for some $\beta \in \mathbb{R}$, then $r(L) \le \beta$. If there exists $v \in K \setminus \{0\}$ such that $Lv \ge \alpha v$ for some $\alpha \in \mathbb{R}$, then $r(L) \ge \alpha$.

Proof. Define $u \in K$ by $u(t) = 1\ \forall t \in Q$. If $f \in X$ and $\|f\| \leq 1$, then $-u \leq f \leq u$, so $-L^k u \leq L^k f \leq L^k u$. It follows that $\|L^k f\| \leq \|L^k u\|$ and this implies $\|L^k\| = \|L^k u\|$ and $r(L) = \lim_{k\to\infty} \|L^k\|^{1/k} = \lim_{k\to\infty} \|L^k u\|^{1/k}$.

If $w \in \text{int}(K)$, there exist positive constants c and d such that $cw \leq u \leq dw$, so, for all positive integers k,

$$cL^k w \leq L^k u \leq dL^k w \text{ and } c\|L^k w\| \leq \|L^k u\| \leq d\|L^k w\|.$$

Taking kth roots and letting $k \to \infty$, we obtain $r(L) = \lim_{k\to\infty} \|L^k w\|^{1/k}$. However, if $Lw \leq \beta w$, $L^k w \leq \beta^k w$, so $r(L) \leq \lim_{k\to\infty} \|\beta^k w\|^{1/k} = \beta$. If $Lv \geq \alpha v$ for some $v \in K \setminus \{0\}$, then $L^k v \geq \alpha^k v$ for all positive integers k and $\|L^k\|\|v\| \geq \alpha^k \|v\|$. Taking kth roots and letting $k \to \infty$, we find that $r(L) \geq \alpha$. □

Note that if we take $Q = \{1, 2, \ldots, N\}$ and identify $C_{\mathbb{R}}(Q)$ with column vectors in $\mathbb{R}^N$, Lemma 2.2 gives results concerning $r(L)$, where $L : \mathbb{R}^N \to \mathbb{R}^N$ is an $N \times N$ matrix with nonnegative entries, or, more abstractly, a linear map which takes the cone of vectors x with nonnegative entries into itself.

Lemma 2.2 is a special case of much more general results concerning order-preserving, homogeneous cone mappings: see [30] and also Lemma 2.2 in [32] and Theorem 2.2 in [34]. In the important special case that L is given by an $N \times N$ matrix with non-negative entries, Lemma 2.2 can also be derived from standard results in [39] concerning nonnegative matrices. A simple proof in the matrix case we consider here can also be found in Lemma 2.2 in [13].

Our next lemma is also a well-known result. Because it follows easily from Lemma 2.2, we leave the proof to the reader.

Lemma 2.3. *Let notation be as in Lemma 2.2. Suppose that $L_j : X \to X$, $j = 1, 2$, are bounded linear maps such that $L_j(K) \subset K$ and $L_1(f) \leq L_2(f)$ for all $f \in K$. Then it follows that $r(L_1) \leq r(L_2)$. If there exists $v \in \text{int}(K)$ with $Lv = \lambda v$, then $r(L) = \lambda$.*

3. Iterated Function Systems Associated to Complex Continued Fractions

3.1. The Problems

Throughout this section we shall always write

$$D := \{(x, y) \in \mathbb{R}^2 : (x - 1/2)^2 + y^2 \leq 1/4\}$$

and U will always denote a bounded, *mildly regular* open subset of $\mathbb{R}^2$ such that $\text{int}(D) \subset U$ and $x > 0$ for all $(x, y) \in U$, while H will denote $\{(x, y) \in U : y > 0\}$. By writing $z = x + iy$, we can consider D, H, and U as subsets of the complex plane. If $S \subset \mathbb{R}^2$, we shall use the identification of $\mathbb{R}^2$ with $\mathbb{C}$ and say that S is symmetric under conjugation if $S = \{\bar{z} : z \in S\}$, where $\bar{z}$ denotes the complex conjugate of z.

In this section, $\mathcal{B}$ will always denote a finite or countable infinite subset of $\{w \in \mathbb{C} := \mathbb{R}^2 : \text{Re}(w) \geq 1\}$, and for $b \in \mathcal{B}$, θ_b will denote the Möbius transform $z \mapsto 1/(z+b) := \theta_b(z)$. If $G := \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$, the reader can check that for all $b \in \mathcal{B}$, $\theta_b(G) \subset D \setminus \{0\}$; and if $b, c \in \mathcal{B}$ satisfy $\text{Re}(b) \geq \gamma \geq 1$

and $\mathrm{Re}(c) \geq \gamma \geq 1$, then $\theta_b \circ \theta_c : G \mapsto D\setminus\{0\}$ is a Lipschitz map (with respect to the Euclidean metric) with Lipschitz constant $\mathrm{Lip}(\theta_b \circ \theta_c) \leq (\gamma^2+1)^{-2}$ (see Lemma 5.1 below). We shall always write $I_1 := \{b = m + n\mathrm{i} : m \in \mathbb{N}, n \in \mathbb{Z}\}$ and the case that $\mathcal{B} \subset I_1$ will be of particular interest.

We shall denote by $C_{\mathbb{C}}(\bar{U})$ (respectively, $C_{\mathbb{R}}(\bar{U})$) the Banach space of continuous maps $f : \bar{U} \to \mathbb{C}$ (respectively, $f : \bar{U} \to \mathbb{R}$) with $\|f\| = \max\{|f(z)| : z \in \bar{U}\}$. (Note that $\bar{U}$ will always denote the closure of U and *not* the image of U under complex conjugation.) If $\mathcal{B}$ is a finite set and $s > 0$, one can define a bounded, complex linear map $L_s : C_{\mathbb{C}}(\bar{U}) \to C_{\mathbb{C}}(\bar{U})$ by

$$(L_s f)(z) = \sum_{b\in\mathcal{B}} \left|\frac{d}{dz}\theta_b(z)\right|^s f(\theta_b(z)) = \sum_{b\in\mathcal{B}} \frac{f(\theta_b(z))}{|z+b|^{2s}}. \tag{3.1}$$

Equation (3.1) also defines a bounded, real linear map of $C_{\mathbb{R}}(\bar{U}) \to C_{\mathbb{R}}(\bar{U})$, which (abusing notation) we shall also denote by L_s. We shall denote by $\sigma(L_s)$ the spectrum of $L_s : C_{\mathbb{C}}(\bar{U}) \to C_{\mathbb{C}}(\bar{U})$.

If $\mathcal{B}$ is infinite, one can prove (see Section 5 of [40] and [44]) that if, for some $s > 0$, the infinite series $\sum_{b\in\mathcal{B}}[1/|b|^{2s}]$ converges, then $\sum_{b\in\mathcal{B}}[1/|z+b|^{2s}]$ converges for all $z \in \bar{U}$ and gives a continuous function on $\bar{U}$. It then follows with the aid of Dini's theorem that L_s given by (3.1) defines a bounded linear map of $C_{\mathbb{C}}(\bar{U})$ to itself. If we define $\tau = \tau(\mathcal{B}) := \inf\{s > 0 : \sum_{b\in\mathcal{B}}[1/|b|^{2s}] < \infty\}$ (where we allow $\tau(\mathcal{B}) = \infty$), it follows from the above remarks that for all $s > \tau(\mathcal{B})$, L_s gives a bounded linear map of $C_{\mathbb{C}}(\bar{U})$ to itself. If $s = \tau$, it may or may not happen that $\sum_{b\in\mathcal{B}}[1/|b|^{2s}] < \infty$. In any event, an elementary calculus argument shows that if $s > 1$, $\sum_{b\in\mathcal{B}}[1/|b|^{2s}] < \infty$.

Our goal in the section is to describe how to obtain rigorous upper and lower bounds for $r(L_s)$, the spectral radius of the operator L_s in (3.1), and then to indicate how such bounds enable us to rigorously estimate the Hausdorff dimension of some interesting sets. To avoid interrupting the narrative flow, we first list some results which we shall need, but whose proofs will be deferred to Sects. 4 and 5. If $\alpha \geq 0$, $R > 0$, and $\mathcal{B}$ is as before, we define

$$\mathcal{B}_R = \{b \in \mathcal{B} : |b| \leq R\} \qquad \text{and} \qquad \mathcal{B}'_R = \{b \in \mathcal{B} : |b| > R\}.$$

If $\mathcal{B}$ is finite, we shall usually take $R \geq \sup\{|b| : b \in \mathcal{B}\}$, so $\mathcal{B}_R = \mathcal{B}$. We define $L_{s,R,\alpha} : C_{\mathbb{C}}(\bar{U}) \to C_{\mathbb{C}}(\bar{U})$ by

$$(L_{s,R,\alpha} f)(z) = \sum_{b\in\mathcal{B}_R} \frac{f(\theta_b(z))}{|z+b|^{2s}} + \alpha f(0). \tag{3.2}$$

Theorem 3.1. *Assume that $\mathcal{B}$ is finite and $\mathrm{Re}(b) \geq \gamma \geq 1$ for all $b \in \mathcal{B}$. For each $s \geq 0$, there exists a unique (to within scalar multiples) strictly positive continuous eigenfunction $w_s \in C_{\mathbb{R}}(\bar{U})$ with positive eigenvalue $r(L_{s,R,\alpha})$ defined by $r(L_{s,R,\alpha}) := \lim_{k\to\infty} \|L^k_{s,R,\alpha}\|^{1/k}$. (Of course w_s also depends on α and R, but we view α and R as fixed and omit the dependence in our notation.) If $\mathcal{B}$ and U are symmetric under conjugation, then $w_s(\bar{z}) = w_s(z)$ for all $z \in \bar{U}$. In general, identifying $(x, y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$, $w_s(x, y)$ is C^∞ on $\bar{U}$ and the following estimates hold.*

$$w_s(z_0) \leq w_s(z_1)\exp[(\sqrt{5}s/\gamma)|z_1 - z_0|], \quad z_0, z_1 \in \bar{U}, \tag{3.3}$$

$$w_s(x_1,y) \ge w_s(x_2,y) \ge w_s(x_1,y)\exp[(-2s/\gamma)(x_2-x_1)],$$
$$0 \le x_1 \le x_2, \quad (x_1,y),(x_2,y) \in \bar{U}, \tag{3.4}$$
$$w_s(x,y_1) \le w_s(x,y_2)\exp[(s/\gamma)|y_1-y_2|],$$
$$(x,y_1),(x,y_2) \in \bar{U}, \tag{3.5}$$
$$-\frac{s}{4\gamma^2(s+1)} w_s(x,y) \le D_{xx} w_s(x,y) \le \frac{2s(2s+1)}{\gamma^2} w_s(x,y), \tag{3.6}$$
$$-\frac{2s}{\gamma^2} w_s(x,y) \le D_{yy} w_s(x,y) \le \frac{2s(2s+1)}{4\gamma^2} w_s(x,y). \tag{3.7}$$

Proof. As mentioned above, the proof of this theorem is contained in a series of results to be established in Sects. 4 and 5. We discuss here how these later results fit together to establish this theorem. The operator $L_{s,R,\alpha}$ can be considered as a bounded linear map of $C^m_{\mathbb{C}}(\bar{U})$ to itself for all integers $m \ge 0$, and conditions (H4.1) and (H4.2) in Sect. 4 are clearly satisfied. Keeping in mind that the constant map $\psi(z) := 0$ is a contraction mapping and using Lemma 5.1, one can also see that $L_{s,R,\alpha} : C^m_{\mathbb{C}}(\bar{U}) \to C^m_{\mathbb{C}}(\bar{U})$ also satisfies condition (H4.3). If $\Lambda_{s,m}$ denotes $L_{s,R,\alpha}$ considered as a map of $C^m_{\mathbb{C}}(\bar{U})$ to itself, and $r_{s,m}$ denotes the spectral radius of $\Lambda_{s,m}$, Theorem 4.1 now implies that for $m \ge 1$, $\Lambda_{s,m}$ has a unique, normalized, strictly positive eigenfunction $w_{s,m}$ with eigenvalue $r_{s,m}$. By using Lemma 2.2 and the strictly positive eigenfunction $w_{s,m}$, we then see that $r_{s,m} = r_{s,0}$ for all $m \ge 1$; and by the uniqueness of $w_{s,m}$, $w_{s,m} = w_{s,1}$ for all $m \ge 1$ and $w_{s,1} := w_s$ is C^∞. This gives the first part of Theorem 3.1. The proof that $w_s(\bar{z}) = w_s(z)$ for all $z \in \bar{U}$ when $\bar{U}$ and $\mathcal{B}$ are symmetric under conjugation is given in Corollary 5.9. Corollary 5.9 also gives the proof of equations (3.3)– (3.7). □

Theorem 3.2. *Assume that $\mathcal{B}$ is infinite and $s > 0$ satisfies $\sum_{b\in\mathcal{B}}[1/|b|^{2s}] < \infty$. Then L_s has a unique (to within scalar multiples) strictly positive eigenfunction $v_s \in C_{\mathbb{R}}(\bar{U})$ with positive eigenvalue $r(L_s)$. This eigenfunction is Lipschitz and satisfies* (3.3), (3.4), *and* (3.5). *If $\mathcal{B}$ and U are symmetric under conjugation, then $v_s(\bar{z}) = v_s(z)$ for all $z \in U$.*

A proof of Theorem 3.2 is given in Theorem 5.10. Several of the results of this theorem can also be found in [37].

Theorem 3.3. *Let assumptions and notation be as in Theorem 3.2 and assume that $R > 2$. Then there exist (see Theorems 5.12 and 5.13) real numbers $\eta_{s,R} \ge 0$ and $\delta_{s,R} > 0$ such that for all $z \in \bar{U}$,*

$$\eta_{s,R} v_s(0) \le \sum_{b\in\mathcal{B},|b|>R} \frac{v_s(\theta_b(z))}{|z+b|^{2s}} \le \delta_{s,R} v_s(0).$$

If $\mathcal{B} = I_1$ or $\mathcal{B} = I_2 := \{m + ni : m \in \mathbb{N}, n \in \mathbb{Z}, n < 0\}$ and $s > 1$, explicit estimates for $\eta_{s,R}$ and $\delta_{s,R}$ are given in Theorems 5.12 and 5.13. If $\alpha = \delta_{s,R}$,

$$r(L_s) \le r(L_{s,R,\alpha}); \tag{3.8}$$

and if $\alpha = \eta_{s,R}$,

$$r(L_{s,R,\alpha}) \le r(L_s). \tag{3.9}$$

If $\mathcal{B}$ is finite, we shall usually assume that $|b| \leq R$ for all $b \in \mathcal{B}$ and take $\alpha = 0$. If $\mathcal{B}$ is infinite, we take R large and use (3.8) and (3.9) to estimate $r(L_s)$. In all cases our problem reduces to finding a procedure which gives rigorous upper and lower bounds for operators $L_{s,R,\alpha}$, where $\alpha = \delta_{s,R}$ or $\alpha = \eta_{s,R}$, or $\alpha = 0$.

If $\mathcal{B}$ and U are symmetric under conjugation, let H be as defined at the beginning of this section and let $\bar{H}$ denote the closure of H. Let $Y = \{f \in C_{\mathbb{C}}(\bar{U}) : f(z) = f(\bar{z}), z \in \bar{U}\}$, so Y is a complex Banach space, and one can check that Y is linearly isometric to $C_{\mathbb{C}}(\bar{H})$ by $f \in Y \mapsto f|_{\bar{H}} \in C_{\mathbb{C}}(\bar{H})$ and $g \in C_{\mathbb{C}}(\bar{H}) \mapsto \tilde{g} \in Y$, where $\tilde{g}(z) = g(z)$ if $z \in \bar{H}$ and $\tilde{g}(z) = g(\bar{z})$ if $z \in \bar{U}$ and $z \notin \bar{H}$. In the notation of Theorem 3.2, $w_s \in Y$, and the reader can check that $L_{s,R,\alpha}$ maps Y into Y, Equivalently, $L_{s,R,\alpha}$ can be viewed as a bounded linear map of $C_{\mathbb{C}}(\bar{H})$ to $C_{\mathbb{C}}(\bar{H})$ by defining $f(1/(z+b)) = f(1/(\bar{z}+\bar{b}))$ if $\mathrm{Im}(z+b) \geq 0$ and $f(1/(z+b)) = f(1/(z+b))$ if $\mathrm{Im}(z+b) \leq 0$. This observation will simplify the numerical analysis in later examples.

If $\mathrm{Im}(b) \leq -1$ for all $b \in \mathcal{B}$ (but without the assumption that $\mathcal{B}$ and U are symmetric under conjugation) and if $\mathrm{Im}(z) \leq 1$ for all $z \in \bar{U}$, one can easily verify that $\theta_b(z) \in \bar{H}$ for all $b \in \mathcal{B}$ and $z \in \bar{U}$. Thus, again in this case one can consider $L_{s,R,\alpha}$ as a map of $C_{\mathbb{C}}(\bar{H})$ to itself, which again will simplify the numerical analysis.

We now briefly discuss the connection of Theorems 3.1–3.3 to the problem of computing the Hausdorff dimension of certain sets.

If $\mathcal{B} \subset I_1$, let $\mathcal{B}_\infty = \{\omega = (b_1, \ldots, b_k, \ldots) : b_j \in \mathcal{B}\ \forall j \geq 1\}$. Given $z \in D$ and $\omega = (b_1, \ldots, b_k, \ldots) \in \mathcal{B}_\infty$, one can prove (e.g., see [37]) that $\lim_{k\to\infty}(\theta_{b_1} \circ \theta_{b_2} \circ \cdots \circ \theta_{b_k})(z) := \pi(\omega) \in D$ exists and is independent of z. Define $C = \{\pi(\omega) : \omega \in \mathcal{B}_\infty\}$. It is not hard to prove that $C = \cup_{b\in\mathcal{B}}\theta_b(C)$. In general C is not compact, but if $\mathcal{B}$ is finite, C is compact and is the unique compact, nonempty set C such that $C = \cup_{b\in\mathcal{B}}\theta_b(C)$ ([11] and [22]). We shall call C the invariant set associated to $\mathcal{B}$.

Theorem 3.4. (See [37] or Section 5 of [44]) *Let $\mathcal{B}$ be a subset of I_1, let $L_s : C_{\mathbb{R}}(\bar{U}) \to C_{\mathbb{R}}(\bar{U})$ be defined by (3.1) for $s > \tau(\mathcal{B})$, and let C be the invariant set associated to $\mathcal{B}$. The Hausdorff dimension s_* of C is given by $s_* = \inf\{s > 0 : r(L_s) = \lambda_s < 1\}$ and $r(L_{s_*}) = 1$ if $\mathcal{B}$ is finite or L_{s_*} is defined. The map $s \mapsto \lambda_s$ is a continuous, strictly decreasing function for $s > \tau(\mathcal{B})$.*

In all examples which we shall consider, L_s is a bounded linear map of $C_{\mathbb{C}}(U) \to C_{\mathbb{C}}(U)$ for $s = s_*$ and $r(L_{s_*}) = 1$.

Theorems 3.1–3.4 reduce the problem of estimating the Hausdorff dimension of the invariant set C for $\mathcal{B} \subset I_1$ to the problem of estimating the value of s for which $r(L_s) = 1$. If $\mathcal{B}$ is finite, we have to estimate $r(L_{s,R,\alpha})$ for $\alpha = 0$. If $\mathcal{B}$ is infinite, Theorem 3.3 implies that we need a lower bound for $r(L_{s,R,\alpha})$ for $\alpha = \eta_{s,R}$ and an upper bound for $r(L_{s,R,\alpha})$ for $\alpha = \delta_{s,R}$.

If $\mathcal{B} = I_1$, it was stated in [36] that the Hausdorff dimension of the associated invariant set C is ≤ 1.885 and in [45], it was shown that the Hausdorff dimension of the set C is ≥ 1.78. We shall give much sharper

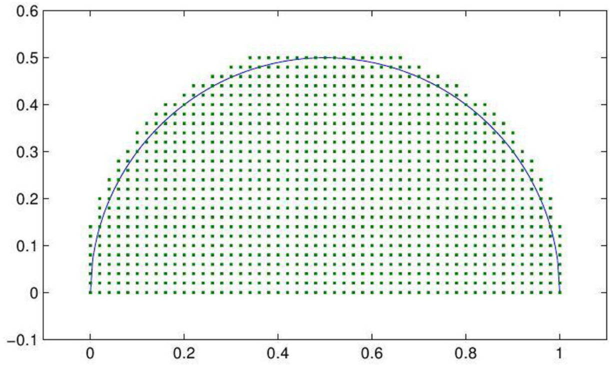

FIGURE 1. Domain D_+ and mesh domain $D_{+,h}$

estimates below. We shall also give estimates for the Hausdorff dimension of the associated invariant set of $\mathcal{B} \subset I_1$ for some other choices of $\mathcal{B}$, e.g.,

$$\mathcal{B} = I_2 := \{b = m + n\mathrm{i} : m \in \mathbb{N}, -n \in \mathbb{N}\},$$
$$\mathcal{B} = I_3 := \{b = m + n\mathrm{i} : m \in \{1, 2\}, n \in \{0, \pm 1, \pm 2\}\}.$$

This is a feasibility study, so we restrict attention to these examples, but our approach applies to general sets $\mathcal{B} \subset I_1$; and in fact invariant sets for many other *iterated function systems* can be handled by similar methods.

3.2. Numerical Method

Let $N > 0$ be an even integer, $h := 1/N$, and let D, U, and H be as in Sect. 3.1. Define $D_+ = \{(x, y) \in D : y \geq 0\}$. We consider *mesh points* of the form (jh, kh), where $j \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$. Each mesh point $(x_j, y_k) = (jh, kh)$ defines a closed *mesh square* R_{jk} with vertices (x_j, y_k), (x_{j+1}, y_k), (x_j, y_{k+1}), and (x_{j+1}, y_{k+1}). If D_h (respectively, $D_{+,h}$) is a finite union of mesh squares and $D_h \supset D$ (respectively $D_{+,h} \supset D_+$), D_h will be called a *mesh domain* for D (respectively, a *mesh domain* for D_+). We could choose $D_{+,h} = [0, 1] \times [0, 1/2]$, but that choice would add unknowns we do not use. Thus we shall usually take D_h (respectively, $D_{+,h}$) to be the union of squares $R_{j,k}$ which have nonempty intersection with the interior of D (respectively, D_+). The domain D_+ and a mesh domain $D_{+,h}$ are illustrated in Figure 1.

The mesh domains D_h and $D_{+,h}$ correspond to sets $\bar{U}$ and $\bar{H}$ in Sect. 3.1. If D and $\mathcal{B}$ are symmetric under conjugation or if $\mathrm{Im}(b) \leq -1$ for all $b \in \mathcal{B}$, the observations in Sect. 3.1 show that we can restrict attention to D_+ and $D_{+,h}$ instead of the full sets D and D_h. Indeed, this will be the case for the invariant sets associated to I_1, I_2, and I_3. We also note that in the case $\mathcal{B} = I_3$, there is a smaller domain $C \subset D$, symmetric under conjugation, such that $\theta_b(C) \subset C \setminus \{0\}$ for $b \in \mathcal{B}$. Although we have not done so, we could have reduced the size of the approximate problem by using a mesh domain C_h for C.

If D_h is as above, we take $\bar{U} = D_h$ and we assume that $0 \leq x \leq 1$ and $|y| < 1$ for all $(x, y) \in \bar{U}$. Given a set $\mathcal{B} \subseteq I_1$ and $s > 0$, we assume that

$s > \tau(\mathcal{B})$ (so $\sum_{b\in\mathcal{B}}(1/|b|^{2s}) < \infty$). If $\mathcal{B}$ is finite, we assume that $R \geq |b|$ for all $b \in \mathcal{B}$ and define $L_s := L_{s,R,\alpha}$ with $\alpha = 0$. If $\mathcal{B}$ is infinite, we assume for the moment that we have found $\eta_{s,R} \geq 0$ and $\delta_{s,R} > 0$ satisfying (3.8) and (3.9). For $\alpha = \eta_{s,R}$, we define $L_{s,R-} = L_{s,R,\alpha}$ and for $\alpha = \delta_{s,R}$, we define $L_{s,R+} = L_{s,R,\alpha}$ (compare (3.2)); we recall that Theorem 3.3 implies that

$$r(L_{s,R-}) \leq r(L_s) \leq r(L_{s,R+}).$$

In all cases, we have an operator $L_{s,R,\alpha}$ where $\alpha \geq 0$ and $R > 2$. Theorem 3.1 implies that $L_{s,R,\alpha}$ has a unique (to within scalar multiples) strictly positive eigenfunction w_s on $\bar{U} = D_h$ which has (assuming $\alpha > 0$ or $\mathcal{B}_R \neq \emptyset$) eigenvalue $r(L_{s,R,\alpha}) > 0$. The eigenfunction w_s is C^∞ and satisfies (3.3)–(3.7). If $\mathcal{B}$ is symmetric under conjugation, $w_s(\bar{z}) = w_s(z)$ for all $z \in D_h$.

We shall now describe how to find rigorous upper and lower bounds for $r(L_{s,R,\alpha})$, where $\alpha \geq 0$ or $\mathcal{B}_R \neq \emptyset$. After estimating $\eta_{s,R}$ and $\delta_{s,R}$, this will yield rigorous upper and lower bounds for $r(L_s)$. Our approach is to approximate w_s by a continuous, piecewise bilinear function, i.e., w_s will be bilinear on each mesh square $R_{j,k}$ of the mesh domain D_h. As noted in Sect. 3.1, we shall be able to work on $D_{+,h}$ in our particular examples.

More precisely, for fixed R and α, our goal is to define nonnegative, square matrices A_s and B_s such that

$$r(A_s) \leq r(L_s) \leq r(B_s), \quad s > \tau(\mathcal{B}).$$

If s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, then s_* is the Hausdorff dimension of the invariant set associated with $\mathcal{B}$. If we can find a number s_1 such that $r(B_{s_1}) \leq 1$, then $r(L_{s_1}) \leq r(B_{s_1}) \leq 1$, and we can conclude that $s_* \leq s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \geq 1$, then $r(L_{s_2}) \geq r(A_{s_2}) \geq 1$, and we can conclude that $s_* \geq s_2$. By choosing the mesh size h to be sufficiently small, we can make $s_1 - s_2$ small, providing a good estimate for s_*.

Before describing how to construct the matrices A_s and B_s, we need to recall some standard results about bilinear interpolation. On the mesh square

$$R_{k,l} = \{(x,y) : x_k \leq x \leq x_{k+1}, y_l \leq y \leq y_{l+1}\},$$

where $x_{k+1} - x_k = y_{l+1} - y_l = h$, the bilinear interpolant $f^I(x,y)$ of a function $f(x,y)$ is given by:

$$\begin{aligned} f^I(x,y) &= \left[\frac{x_{k+1}-x}{h}\right]\left[\frac{y_{l+1}-y}{h}\right] f(x_k,y_l) + \left[\frac{x-x_k}{h}\right]\left[\frac{y_{l+1}-y}{h}\right] f(x_{k+1},y_l) \\ &\quad + \left[\frac{x_{k+1}-x}{h}\right]\left[\frac{y-y_l}{h}\right] f(x_k,y_{l+1}) + \left[\frac{x-x_k}{h}\right]\left[\frac{y-y_l}{h}\right] f(x_{k+1},y_{l+1}). \end{aligned}$$

The error in bilinear interpolation satisfies for all $(x,y) \in R_{k,l}$ and some points (a_k,b_l) and $(c_k,d_l) \in R_{k,l}$,

$$\begin{aligned} f^I(x,y) - f(x,y) = 1/2)\Big[&(x_{k+1}-x)(x-x_k)(D_{xx}f)(a_k,b_l) \\ &+ (y_{l+1}-y)(y-y_l)(D_{yy}f)(c_k,d_l)\Big]. \end{aligned}$$

For $z = x + \mathrm{i}y$, let $f(x, y) = w_s(\theta_b(z))$. Further let $z_{k,l} = x_k + \mathrm{i}y_l$. If $(\tilde{x}, \tilde{y}) = (\operatorname{Re}\theta_b(z), \operatorname{Im}\theta_b(z)) \in R_{k,l}$, (which we will sometimes abbreviate by $\theta_b(z) \in R_{k,l}$), we get

$$\begin{aligned} &w_s^I(\theta_b(z)) \\ &\quad= \Big[\frac{x_{k+1}-\tilde{x}}{h}\Big]\Big[\frac{y_{l+1}-\tilde{y}}{h}\Big] w_s(z_{k,l}) + \Big[\frac{\tilde{x}-x_k}{h}\Big]\Big[\frac{y_{l+1}-\tilde{y}}{h}\Big] w_s(z_{k+1,l}) \\ &\qquad+ \Big[\frac{x_{k+1}-\tilde{x}}{h}\Big]\Big[\frac{\tilde{y}-y_l}{h}\Big] w_s(z_{k,l+1}) + \Big[\frac{\tilde{x}-x_k}{h}\Big]\Big[\frac{\tilde{y}-y_l}{h}\Big] w_s(z_{k+1,l+1}). \end{aligned}$$

Defining $\Psi_b(z) = 1/(\bar{z} + \bar{b})$, we have an analogous formula for $w_s^I(\Psi_b(z))$, with

$$(\tilde{x}, \tilde{y}) = (\operatorname{Re}\Psi_b(z), \operatorname{Im}\Psi_b(z)).$$

We next use inequalities (3.3)–(3.7) to obtain bounds on the interpolation error. By (3.6) and (3.7), we find for $\theta_b(z) = \tilde{x} + \mathrm{i}\tilde{y}$, where $(\tilde{x}, \tilde{y}) \in R_{k,l}$,

$$\begin{aligned} &-\Big[\frac{s}{8\gamma^2(s+1)} + \frac{s}{\gamma^2}\Big] \left([x_{k+1}-\tilde{x}][\tilde{x}-x_k] w_s(a_k, b_l)\right. \\ &\qquad \left. + [y_{l+1}-\tilde{y}][\tilde{y}-y_l] w_s(c_k, d_l)\right) \le w_s^I(\theta_b(z)) - w_s(\theta_b(z)) \\ &\quad\le \frac{s(2s+1)}{\gamma^2}\left([x_{k+1}-\tilde{x}][\tilde{x}-x_k] w_s(a_k, b_l) + [y_{l+1}-\tilde{y}][\tilde{y}-y_l] w_s(c_k, d_l)\right). \end{aligned}$$

Applying (3.3), we then obtain

$$\begin{aligned} &-\frac{s}{\gamma^2}\Big[\frac{9+8s}{8(s+1)}\Big]\left([x_{k+1}-\tilde{x}][\tilde{x}-x_k] + [y_{l+1}-\tilde{y}][\tilde{y}-y_l]\right) \\ &\qquad \cdot \exp\Big(\frac{\sqrt{10}sh}{\gamma}\Big) w_s^I(\theta_b(z)) \le w_s^I(\theta_b(z)) - w_s(\theta_b(z)) \\ &\quad\le \frac{s(2s+1)}{\gamma^2}\left([x_{k+1}-\tilde{x}][\tilde{x}-x_k] + [y_{l+1}-\tilde{y}][\tilde{y}-y_l]\right) \\ &\qquad \cdot \exp\Big(\frac{\sqrt{10}sh}{\gamma}\Big) w_s^I(\theta_b(z)) \le w_s^I(\theta_b(z)) - w_s(\theta_b(z)) \\ &\quad\le \frac{s(2s+1)}{\gamma^2}\left([x_{k+1}-\tilde{x}][\tilde{x}-x_k] + [y_{l+1}-\tilde{y}][\tilde{y}-y_l]\right) \\ &\qquad \cdot \exp\Big(\frac{\sqrt{10}sh}{\gamma}\Big) w_s^I(\theta_b(z)). \end{aligned}$$

since any point in $R_{k,l}$ is within $\sqrt{2}h$ of each of the four corners of the square $R_{k,l}$. An analogous result holds for $w_s(\Psi_b(z))$.

Using this estimate, we have precise upper and lower bounds on the error in the mesh square $R_{k,l}$ that only depend on the function values of w_s at the four corners of the square and the value of b. Letting

$$\begin{aligned} &\mathrm{err}_b^1(\theta_b(z)) \\ &\quad= \Big([x_{k+1}-\tilde{x}][\tilde{x}-x_k] + [y_{l+1}-\tilde{y}][\tilde{y}-y_l]\Big)\frac{s(2s+1)}{\gamma^2}\exp(\sqrt{10}sh/\gamma), \\ &\mathrm{err}_b^2(\theta_b(z)) \end{aligned}$$

$$= \Big([x_{k+1}-\tilde{x}][\tilde{x}-x_k]+[y_{l+1}-\tilde{y}][\tilde{y}-y_l]\Big)\frac{s}{\gamma^2}\left[\frac{9+8s}{8+8s}\right]\exp(\sqrt{10}sh/\gamma),$$

(where again $\theta_b(z)=\tilde{x}+\mathrm{i}\tilde{y}$), we have for each mesh point $z_{i,j}=x_i+\mathrm{i}y_j$, with $\theta_b(z_{i,j})\in R_{k,l}$,

$$[1-\mathrm{err}_b^1(z_{i,j})]w_s^I(\theta_b(z_{i,j}))\le w_s(\theta_b(z_{i,j}))\le[1+\mathrm{err}_b^2(z_{i,j})]w_s^I(\theta_b(z_{i,j})).$$

Again, the analogous result holds for $w_s(\Psi_b(z))$.

To obtain the upper and lower matrices, we first note that for each mesh point $z_{i,j}$,

$$\begin{aligned}\alpha w_s(0)&+\sum_{b\in\mathcal{B}_R}\frac{1}{|z_{i,j}+b|^{2s}}[1-\mathrm{err}_b^1(z_{i,j})]w_s^I(\theta_b(z_{i,j}))\\&\le\sum_{b\in\mathcal{B}_R}\frac{1}{|z_{i,j}+b|^{2s}}w_s(\theta_b(z_{i,j}))+\alpha w_s(0)\\&\le\sum_{b\in\mathcal{B}_R}\frac{1}{|z_{i,j}+b|^{2s}}[1+\mathrm{err}_b^2(z_{i,j})]w_s^I(\theta_b(z_{i,j}))+\alpha w_s(0).\end{aligned}$$

Motivated by the above inequality, we now define matrices A_s and B_s which have nonnegative entries and satisfy the property that $r(A_s)\le r(L_s)\le r(B_s)$. For clarity, we do this in several steps. For f a continuous, piecewise bilinear function defined on the mesh domain D_h, we first define operators $\boldsymbol{A}_s$ and $\boldsymbol{B}_s$ (which also depend on α) by:

$$(\boldsymbol{A}_s f)(z_{i,j})=\sum_{b\in\mathcal{B}_R}\frac{1}{|z_{i,j}+b|^{2s}}[1-\mathrm{err}_b^1(z_{i,j})]f(\theta_b(z_{i,j}))+\alpha f(0), \tag{3.10}$$

$$(\boldsymbol{B}_s f)(z_{i,j})=\sum_{b\in\mathcal{B}_R}\frac{1}{|z_{i,j}+b|^{2s}}[1+\mathrm{err}_b^2(z_{i,j})]f(\theta_b(z_{i,j}))+\alpha f(0), \tag{3.11}$$

where $z_{i,j}$ is a mesh point in D_h. In the above, if $(\tilde{x},\tilde{y})=(\mathrm{Re}\,\theta_b(z),\mathrm{Im}\,\theta_b(z))\in R_{k,l}$, then, using bilinearity,

$$\begin{aligned}f(\theta_b(z))=&\left[\frac{x_{k+1}-\tilde{x}}{h}\right]\left[\frac{y_{l+1}-\tilde{y}}{h}\right]f(z_{k,l})+\left[\frac{\tilde{x}-x_k}{h}\right]\left[\frac{y_{l+1}-\tilde{y}}{h}\right]f(z_{k+1,l})\\&+\left[\frac{x_{k+1}-\tilde{x}}{h}\right]\left[\frac{\tilde{y}-y_l}{h}\right]f(z_{k,l+1})\\&+\left[\frac{\tilde{x}-x_k}{h}\right]\left[\frac{\tilde{y}-y_l}{h}\right]f(z_{k+1,l+1}).\end{aligned} \tag{3.12}$$

Let $Q=\{z_{i,j}\,:\,z_{i,j}$ is a mesh point of $D_h\}$ and consider the finite dimensional vector space $C_{\mathbb{R}}(Q)$. We can consider f above as an element of $C_{\mathbb{R}}(Q)$, where $f(\theta_b(z))$ is defined by (3.12). If we use (3.12) in (3.10) and (3.11), $\boldsymbol{A}_s$ and $\boldsymbol{B}_s$ define linear maps of $C_{\mathbb{R}}(Q)$ to $C_{\mathbb{R}}(Q)$. Note that since $\mathrm{err}_b^i=O(h^2)$ for $i=1,2$, $\boldsymbol{A}_s(S+)\subset S+$ and $\boldsymbol{B}_s(S+)\subset S+$ for h sufficiently small, where $S+$ denotes the set of nonnegative functions in $C_{\mathbb{R}}(Q)$. If, for fixed $\alpha \geq 0$, we take $f = w_s$, the strictly positive eigenfunction of $L_{s,R,\alpha}$, our construction insures that for all mesh points $z_{i,j} \in D_h$,

$$(\boldsymbol{A}_s w_s)(z_{i,j}) \leq (L_{s,R,\alpha} w_s)(z_{i,j}) = r(L_{s,R,\alpha}) w_s(z_{i,j}) \leq (\boldsymbol{B}_s w_s)(z_{i,j}).$$

Lemma 2.2 now implies that

$$r(\boldsymbol{A}_s) \leq r(L_{s,R,\alpha}) \leq r(\boldsymbol{B}_s). \tag{3.13}$$

If $\mathcal{B}$ is finite, so $\alpha = 0$ and $L_{s,R} = L_s$, (3.13) gives an estimate for $r(L_s)$ in terms of the spectral radii of finite dimensional linear maps $\boldsymbol{A}_s$ and $\boldsymbol{B}_s$. If $\mathcal{B}$ is infinite and $R > 0$ has been chosen and $\eta_{s,R}$ and $\delta_{s,R}$ have been estimated as in Theorems 5.12 and 5.13, we take $\alpha = \eta_{s,R}$ in (3.10) and define $\boldsymbol{A}_s$ as in (3.10) and we obtain, using Theorem 3.3,

$$r(\boldsymbol{A}_s) \leq r(L_{s,R-}) \leq r(L_s). \tag{3.14}$$

Taking $\alpha = \delta_{s,R}$ in (3.11), we define $\boldsymbol{B}_s$ as in (3.11) to obtain

$$r(L_s) \leq r(L_{s,R+}) \leq r(\boldsymbol{B}_s). \tag{3.15}$$

As a practical matter, it remains to describe the linear maps $\boldsymbol{A}_s$ and $\boldsymbol{B}_s$ as matrices. For simplicity, we totally order the elements of Q by the dictionary ordering, i.e., $z_{i,j} < z_{p,q}$ if and only if $i < p$ or if $i = p$ and $j < q$. Then we can identify $f \in C_{\mathbb{R}}(Q)$ with a column vector $(f_1, \ldots, f_k, \ldots, f_n)^T$, where $f(z_{i,j}) := f_k$ if $z_{i,j}$ is the kth element when the mesh points in D_h are ordered as above and n is the total number of mesh points in D_h, Since $f(\theta_b(z))$ is a linear combination of four components of f, the mesh point $z_{i,j}$ will produce row k of the matrix A_s (and similarly for B_s). A more detailed description of this procedure can be found in [13] for a simpler one dimensional problem. Since A_s and B_s are just representations of the linear maps $\boldsymbol{A}_s$ and $\boldsymbol{B}_s$, we can replace $r(\boldsymbol{A}_s)$ by $r(A_s)$ in (3.14) and $r(\boldsymbol{B}_s)$ by $r(B_s)$ in (3.15). Thus, we can restate (3.14) and (3.15) in terms of the spectral radii of the matrices A_s and B_s, which better conforms to the description in Sect. 1:

$$r(A_s) \leq r(L_s) \leq r(B_s).$$

As described in Sect. 1, if s_* denotes the unique value of s such that $r(L_{s_*}) = \lambda_{s_*} = 1$, then s_* is the Hausdorff dimension of the invariant set under study. Hence, if we can find a number s_1 such that $r(B_{s_1}) \leq 1$, then $r(L_{s_1}) \leq r(B_{s_1}) \leq 1$, and we can conclude that $s_* \leq s_1$. Analogously, if we can find a number s_2 such that $r(A_{s_2}) \geq 1$, then $r(L_{s_2}) \geq r(A_{s_2}) \geq 1$, and we can conclude that $s_* \geq s_2$. By choosing the mesh sufficiently fine and both $r(B_{s_1})$ and $r(A_{s_2})$ very close to one, we can make $s_1 - s_2$ small, providing a good estimate for s_*. As noted in Sect. 1, since the map $s \mapsto r(L_{s,R,\alpha})$ is log convex, we can find the desired values of s_1 and s_2 by using a slight modification of the secant method applied to finding zeros of the functions $\log[r(A_{s_2})]$ and $\log[r(B_{s_2})]$. We also note that since the matrices A_s and B_s will have a single dominant eigenvalue, (see Sect. 6 of this paper and Section 7 of [13]), the spectral radius is easily computed by a variant of the power method (in fact, our computer codes simply call the *Matlab* routine `eigs`). Indeed, the same program also gives high order approximations to the strictly positive eigenvectors associated to $r(A_s)$ and $r(B_s)$.

Table 1. Computation of Hausdorff dimension s for several values of h and R (rounded to 5 decimal places)

Set	h	R	Lower s	Upper s
I_1	.02	100	1.85516	1.85608
I_1	.01	100	1.85563	1.85594
I_1	.005	100	1.85574	1.85590
I_1	.02	200	1.85521	1.85604
I_1	.01	200	1.85568	1.85589
I_1	.02	300	1.85522	1.85603
I_2	.02	100	1.48883	1.49010
I_2	.01	100	1.48904	1.49003
I_2	.005	100	1.48909	1.49002
I_2	.02	200	1.48925	1.48985
I_2	.01	200	1.48946	1.48978
I_2	.02	300	1.48933	1.48981
I_3	.02		1.53706	1.53790
I_3	.01		1.53754	1.53774
I_3	.005		1.53765	1.53770

By our remarks above, it only remains to use our estimates for $\eta_{s,R}$ and $\delta_{s,R}$ in (3.8) and (3.9) when $\mathcal{B}$ is infinite, since then we will have the matrices A_s and B_s.

In Table 1, we present the computation of upper and lower bounds for the Hausdorff dimension of the invariant sets associated to $\mathcal{B} = I_1, I_2$, and I_3. In the table, we study the effects of decreasing the mesh size h and increasing the value of R, which corresponds to only including terms in the sum for which $|b| \leq R$. Each row in the table gives upper and lower bounds, and for R fixed, one can see that the lower bounds are increasing and the upper bounds decreasing as h is decreased. Similarly, taking a larger value of R improves the bounds for the same mesh size. Except for possible round off error in these calculations, which we do not expect to affect the results for the number of decimal places shown, our theorems prove that these are in fact upper and lower bounds for the actual Hausdorff dimension.

Remark 3.1. It is important to note that, given s_1 and s_2, B_{s_1} and A_{s_2} are, modulo roundoff errors in computation, known exactly. Furthermore, our computer program furnishes (purported) strictly positive eigenvectors w_{s_1} for B_{s_1} and u_{s_2} for A_{s_2}, with respective eigenvalues $r(B_{s_1}) < 1$ and $r(A_{s_2}) > 1$. However, we do not need to know whether w_{s_1} and u_{s_2} are actually eigenvectors. It suffices to verify that

$$B_{s_1} w_{s_1} \leq w_{s_1} \qquad \text{and} \qquad A_{s_2} u_{s_2} \geq u_{s_2}, \tag{3.16}$$

since then Lemma 2.2 implies that $r(B_{s_1}) \leq 1$ and $r(A_{s_2}) \geq 1$, and we obtain that $s_2 \leq s_* \leq s_1$. The vectors u_{s_2} and w_{s_1} are given to us exactly by the program. We have verified (3.16) to high accuracy, but we have not used interval arithmetic. If we had used interval arithmetic to calculate B_{s_1}, A_{s_2}, and to verify (3.16), the estimates in Table 1 would be completely rigorous. It is in that sense that we list the following result as a theorem.

Theorem 3.5. *The Hausdorff dimensions of the invariant sets associated to* $\mathcal{B} = I_1$*,* I_2*, and* I_3 *satisfy the bounds*

$$I_1: \quad 1.85574 \le s \le 1.85589, \qquad I_2: \quad 1.48946 \le s \le 1.48978,$$
$$I_3: \quad 1.53765 \le s \le 1.53770.$$

3.3. Higher Order Approximation

Although the theory developed in this paper does not apply to higher order piecewise polynomial approximation, since one cannot guarantee that the approximate matrices have nonnegative entries, we also report in Tables 2 and 3 the results of higher order piecewise polynomial approximation to demonstrate the promise of this approach. In this case, we only provide the results for the approximate matrix, which does not contain any corrections for the interpolation error.

Since we did not have an exact solution for the problem corresponding to the set I_3, we cannot compare the actual errors. However, assuming the last entry in Table 2 gives the most accurate approximation, we see that the third entry using piecewise cubics is accurate to 10 decimal places, which is a significant improvement over the last entry for linear approximation, which only produces 5 correct digits after the decimal point. This is consistent with the theory of approximation of smooth functions by piecewise polynomials, which shows that the convergence rate grows as the degree of the polynomials is increased. In the computations shown using higher order piecewise polynomials, to get a fair comparison, we have adjusted the mesh sizes so that the results for different degree piecewise polynomials will have approximately the same number of degrees of freedom (DOF).

Table 2. Computation of Hausdorff dimension s of the set I_3 using higher order piecewise polynomials

Degree	h	# DOF	s
1	0.02	1098	1.537729111247678
1	0.01	4165	1.537694920731214
1	0.005	16201	1.537686565250360
2	0.041667	1041	1.537683708302400
2	0.020833	3913	1.537683729607203
2	0.010417	15089	1.537683732415111
3	0.0625	1081	1.537683753797206
3	0.03125	3997	1.537683734167568
3	0.015625	15283	1.537683732983929
3	0.0078125	59545	1.537683732912027

In a future paper we hope to prove that rigorous upper and lower bounds for the Hausdorff dimension can also be obtained when higher order piecewise polynomial approximations are used.

3.4. A Special Example with a Known Solution

To further test the algorithm, especially using higher order piecewise polynomials, we constructed a special example where the exact solution is known. More specifically, we considered the operator

$$(L_s(f))(z) = \sum_{b \in \mathcal{B}} g_b^s(z) f(\theta_b(z)),$$

where $\mathcal{B} = \{1 \pm \mathrm{i}, 2 \pm \mathrm{i}, 3 \pm \mathrm{i}\}$ and

$$g_b(z) = \frac{1}{6} \left| \frac{z+b+1}{z+b} \right|^2 \left| \frac{1}{z+1} \right|^2.$$

This example is constructed so that $f(z) = |1/(z+1)|^2$ is an eigenfunction of L_1 with eigenvalue $\lambda = 1$ for $s = 1$. In Table 3, we present the results of approximations using different values of h and different degree piecewise polynomials.

4. Existence of C^m Positive Eigenfunctions

In this section we shall describe some results concerning existence of C^m positive eigenfunctions for a class of positive (in the sense of order-preserving) linear operators. We shall later indicate how one can often obtain explicit bounds on partial derivatives of the positive eigenfunctions. As noted above, such estimates play a crucial role in our numerical method and therefore in obtaining rigorous estimates of Hausdorff dimension for invariant sets associated with iterated function systems.

The starting point of our analysis is Theorem 5.5 in [43], which we now describe for a simple case. If H is a bounded open subset of $\mathbb{R}^n$ and

TABLE 3. Approximation, using higher order piecewise polynomials, of the number $s = 1$ for which $r(L_s) = 1$ for the special example

Degree	h	# DOF	s
1	0.02	1098	1.000034749616189
1	0.01	4165	1.000010815423902
1	0.005	16201	1.000002596942892
2	0.02	4239	1.000000016815596
2	0.01	16357	0.999999997912829
3	0.02	9424	1.000000000610834
4	0.04167	4017	0.999999999999715
4	0.02	16653	0.999999999999925

m is a positive integer, $C^m_{\mathbb{C}}(\bar{H})$ will denote the set of complex-valued C^m maps $f : H \to \mathbb{C}$ such that all partial derivatives $D^\alpha f$ with $|\alpha| \leq m$ extend continuously to $\bar{H}$. (Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with $\alpha_j \geq 0$ for all j, $D_j = \partial/\partial x_j$ for $1 \leq j \leq n$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$), $C^m_{\mathbb{C}}(\bar{H})$ is a complex Banach space with $\|f\| = \sup\{|D^\alpha f(x)| : x \in H, |\alpha| \leq m\}$. Analogously, $C^m_{\mathbb{R}}(\bar{H})$ denotes the corresponding real Banach space of real-valued C^m maps $f : H \to \mathbb{R}$.

We say that H is *mildly regular* if there exist $\eta > 0$ and $M \geq 1$ such that whenever $x, y \in H$ and $\|x - y\| < \eta$, there exists a Lipschitz map $\psi : [0, 1] \to H$ with $\psi(0) = x$, $\psi(1) = y$ and

$$\int_0^1 \|\psi'(t)\|\, dt \leq M\|x - y\|. \tag{4.1}$$

(Here $\|\cdot\|$ denotes any fixed norm on $\mathbb{R}^n$. If the norm is changed, (4.1) remains valid, but with a different constant M.)

Let $\mathcal{B}$ denote a finite index set with $|\mathcal{B}| = p$. For $b \in \mathcal{B}$, we assume

(H4.1) $g_b \in C^m_{\mathbb{R}}(\bar{H})$ for all $b \in \mathcal{B}$ and $g_b(x) > 0$ for all $x \in \bar{H}$ and all $b \in \mathcal{B}$.

(H4.2) $\theta_b : H \to H$ is a C^m map for all $b \in \mathcal{B}$, i.e., if

$\theta_b(x) = (\theta_{b_1}(x), \ldots \theta_{b_n}(x))$, then $\theta_{b_k} \in C^m_{\mathbb{R}}(\bar{H})$ for all $b \in \mathcal{B}$ and for $1 \leq k \leq n$.

In (H4.1) and (H4.2), we always assume that $m \geq 1$.

We define a bounded, complex linear map $\Lambda : C^m_{\mathbb{C}}(\bar{H}) \to C^m_{\mathbb{C}}(\bar{H})$ by

$$(\Lambda(f))(x) = \sum_{b \in B} g_b(x) f(\theta_b(x)). \tag{4.2}$$

Equation (4.2) also defines a bounded real linear map of $C^m_{\mathbb{R}}(\bar{H})$ to itself which we shall also denote by Λ.

For integers $\mu \geq 1$, we define $\mathcal{B}_\mu := \{\omega = (j_1, \ldots j_\mu) : j_k \in \mathcal{B}$ for $1 \leq k \leq \mu\}$. For $\omega = (j_1, \ldots j_\mu) \in \mathcal{B}_\mu$, we define $\omega_\mu = \omega$, $\omega_{\mu-1} = (j_1, \ldots j_{\mu-1})$, $\omega_{\mu-2} = (j_1, \ldots j_{\mu-2})$, $\cdots$, $\omega_1 = j_1$. We define

$$\theta_{\omega_{\mu-k}}(x) = (\theta_{j_{\mu-k}} \circ \theta_{j_{\mu-k-1}} \circ \cdots \circ \theta_{j_1})(x),$$

so

$$\theta_\omega(x) := \theta_{\omega_\mu}(x) = (\theta_{j_\mu} \circ \theta_{j_{\mu-1}} \circ \cdots \circ \theta_{j_1})(x).$$

For $\omega \in \mathcal{B}_\mu$, we define $g_\omega(x)$ inductively by $g_\omega(x) = g_{j_1}(x)$ if $\omega = (j_1) \in \mathcal{B} := \mathcal{B}_1$, $g_\omega(x) = g_{j_2}(\theta_{j_1}(x)) g_{j_1}(x)$ if $\omega = (j_1, j_2) \in \mathcal{B}_2$ and, for $\omega = (j_1, j_2, \ldots j_\mu) \in \mathcal{B}_\mu$,

$$g_\omega(x) = g_{j_\mu}(\theta_{\omega_{\mu-1}}(x)) g_{\omega_{\mu-1}}(x).$$

If is not hard to show (see [3,40,43]) that

$$(\Lambda^\mu(f))(x) = \sum_{\omega \in \mathcal{B}_\mu} g_\omega(x) f(\theta_\omega(x)). \tag{4.3}$$

If Λ and m are as above, we shall let $\sigma(\Lambda) \subset \mathbb{C}$ denote the spectrum of Λ. If all the functions g_j and θ_j are C^N, then we can consider Λ as a bounded linear operator $\Lambda_m : C^m_{\mathbb{C}}(\bar{H}) \to C^m_{\mathbb{C}}(\bar{H})$ for $1 \le m \le N$, but one should note that in general $\sigma(\Lambda_m)$ will depend on m.

To obtain a useful theory for Λ, we need a further crucial assumption. For a given norm $\|\cdot\|$ on $\mathbb{R}^n$, we assume

(H4.3) There exists a positive integer μ and a constant $\kappa < 1$ such that for all $\omega \in \mathcal{B}_\mu$ and all $x, y \in H$,

$$\|\theta_\omega(x) - \theta_\omega(y)\| \le \kappa \|x - y\|.$$

If we define $c = \kappa^{1/\mu} < 1$, it follows from (H4.3) that there exists a constant M such that for all $\omega \in B_\nu$ and all $\nu \ge 1$,

$$\|\theta_\omega(x) - \theta_\omega(y)\| \le M c^\nu \|x - y\| \quad \forall x, y \in H. \tag{4.4}$$

If the norm $\|\cdot\|$ in (4.4) is replaced by a different norm $|\cdot|$, (4.4) remains valid, although with a different constant M. This in turn implies that (H4.3) will also be valid with the same constant κ, with $|\cdot|$ replacing $\|\cdot\|$ and with a possibly different integer μ.

The following theorem is a special case of Theorem 5.5 in [43].

Theorem 4.1. *Let H be a bounded open subset of $\mathbb{R}^n$ and assume that H is mildly regular. Let $X = C^m_{\mathbb{C}}(\bar{H})$ and assume that (H4.1), (H4.2), and (H4.3) are satisfied (where $m \ge 1$ in (H4.1) and (H4.2)) and that $\Lambda : X \to X$ is given by (4.2). If $Y = C_{\mathbb{C}}(\bar{H})$, the Banach space of complex-valued continuous functions $f : \bar{H} \to \mathbb{C}$ and $L : Y \to Y$ is defined by (4.2), then $r(L) = r(\Lambda) > 0$, where $r(L)$ denotes the spectral radius of L and $r(\Lambda)$ denotes the spectral radius of Λ. If $\rho(\Lambda)$ denotes the essential spectral radius of Λ (see [34,40,41,44]), then $\rho(\Lambda) \le c^m r(\Lambda)$ where $c = \kappa^{1/\mu}$ is as in (4.4). There exists $v \in X$ such that $v(x) > 0$ for all $x \in \bar{H}$ and*

$$\Lambda(v) = rv, \qquad r = r(\Lambda).$$

There exists $r_1 < r$ such that if $\xi \in \sigma(\Lambda) \setminus \{r\}$, then $|\xi| \le r_1$; and $r = r(\Lambda)$ is an isolated point of $\sigma(\Lambda)$ and an eigenvalue of algebraic multiplicity 1. If $u \in X$ and $u(x) > 0 \, \forall x \in \bar{H}$, there exists a real number $s_u > 0$ such that

$$\lim_{k \to \infty} \left(\frac{1}{r} \Lambda \right)^k (u) = s_u v, \tag{4.5}$$

where the convergence in (4.5) is in the C^m topology on X.

Remark 4.1. If α is a multi-index with $|\alpha| \le m$, where $m \ge 1$ is as in (H4.1) and (H4.2), it follows from (4.5) that

$$\lim_{k \to \infty} \left(\frac{1}{r} \right)^k D^\alpha \Lambda^k(u) = s_u D^\alpha v, \tag{4.6}$$

and

$$\lim_{k \to \infty} \left(\frac{1}{r} \right)^k \Lambda^k(u) = s_u v, \tag{4.7}$$

where the convergence in (4.6) and (4.7) is in the topology of $C_{\mathbb{C}}(\bar{H})$, the Banach space of continuous functions $f : \bar{H} \to \mathbb{C}$.

It follows from (4.6) and (4.7) that for any multi-index α with $|\alpha| \le m$,

$$\lim_{k\to\infty} \frac{(D^\alpha \Lambda^k(u))(x)}{\Lambda^k(u)(x)} = \frac{(D^\alpha(v))(x)}{v(x)}, \tag{4.8}$$

where the convergence in (4.8) is uniform in $x \in \bar{H}$. If we choose $u(x) = 1$ for all $x \in \bar{H}$, it follows from (4.3) that for all multi-indices α with $|\alpha| \le m$, we have

$$\lim_{k\to\infty} \frac{D^\alpha(\sum_{\omega\in B_k} g_\omega(x))}{\sum_{\omega\in B_k} g_\omega(x)} = \frac{D^\alpha v(x)}{v(x)}, \tag{4.9}$$

where the convergence in (4.9) is uniform in $x \in \bar{H}$. We shall use (4.9) in our further work to obtain explicit bounds on $\sup\big\{|D^\alpha v(x)|/v(x) : x \in \bar{H}\big\}$.

Direct analogues of Theorem 5.5 in [43] exist when $\mathcal{B}$ is countable but not finite (e.g., see [37,40]), but such analogues were not stated or proved in [43]. We shall make do here with less precise theorems which we shall prove by an *ad hoc* argument in the next section. We refer to Lemma 5.3 in Section 5 of [44], Theorem 5.3 on p. 86 of [40] and Section 5 of [40] for more information about existence of positive eigenfunctions when $\mathcal{B}$ is infinite.

5. The Case of Möbius Transformations

By working with partial derivatives and using methods like those in Section 5 of [13], it is possible to obtain explicit estimates on partial derivatives of $v_s(x)$ in the generality of Theorem 4.1. However, for reasons of length and in view of the immediate applications in this paper, we shall not treat the general case here and shall now specialize to the case that the mappings $\theta_b(\cdot)$ are given by Möbius transformations which map a given bounded open subset H of $\mathbb{C} := \mathbb{R}^2$ into H. Specifically, throughout this section we shall usually assume:

(H5.1): $\gamma \ge 1$ is a given real number and $\mathcal{B}$ is a finite collection of complex numbers b such that $\mathrm{Re}(b) \ge \gamma$ for all $b \in \mathcal{B}$. For each $b \in \mathcal{B}$, $\theta_b(z) := 1/(z+b)$ for $z \in \mathbb{C} \setminus \{-b\}$.

The assumption in (H5.1) that $\gamma \ge 1$ is only a convenience; and the results of this section can be proved under the weaker assumption that $\gamma > 0$.

For $\gamma > 0$ we define $G_\gamma \in \mathbb{C}$ by

$$G_\gamma = \{z \in \mathbb{C} : |z - 1/(2\gamma)| < 1/(2\gamma)\}. \tag{5.1}$$

It is easy to check that if $w \in \mathbb{C}$ and $\mathrm{Re}(w) > \gamma$, then $(1/w) \in G_\gamma$. It follows that if $\mathrm{Re}(z) > 0$, $b \in \mathbb{C}$ and $\mathrm{Re}(b) \ge \gamma > 0$, then $\theta_b(z) \in \bar{G}_\gamma$. Let H be a bounded, open, mildly regular subset of $\mathbb{C} = \mathbb{R}^2$ such that $H \supset G_\gamma$ and $H \subset \{z : \mathrm{Re}(z) > 0\}$, and let $\mathcal{B}$ denote a finite set of complex numbers such that $\mathrm{Re}(b) \ge \gamma > 0$ for all $b \in \mathcal{B}$. We define a bounded linear map $\Lambda_s : C^m_{\mathbb{C}}(\bar{H}) \to C^m_{\mathbb{C}}(\bar{H})$, where m is a positive integer and $s \ge 0$, by

$$(\Lambda_s(f))(z) = \sum_{b\in\mathcal{B}} \Big|\frac{d}{dz}\theta_b(z)\Big|^s f(\theta_b(z)) := \sum_{b\in\mathcal{B}} \frac{1}{|z+b|^{2s}} f(\theta_b(z)). \tag{5.2}$$

As in Sect. 1, $L_s : C_{\mathbb{C}}(\bar{H}) \to C_{\mathbb{C}}(\bar{H})$ is defined by (5.2). We use different letters to emphasize that $\sigma(\Lambda_s) \ne \sigma(L_s)$, although $r(\Lambda_s) = r(L_s)$.

If all elements of $\mathcal{B}$ are real, we can restrict attention to the real line and, as we shall see, the analysis is much simpler. In this case we abuse notation and take $G_\gamma = (0, 1/\gamma) \subset \mathbb{R}^2$ and $H = (0, a)$, $a \geq 1/\gamma$. For $f \in C^m_{\mathbb{C}}(\bar{H})$ and $x \in \bar{H}$, (5.2) takes the form

$$(\Lambda_s(f))(x) = \sum_{b \in \mathcal{B}} \frac{1}{(x+b)^{2s}} f(\theta_b(x)).$$

If, for $1 \leq j \leq n$, $M_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ is a 2×2 matrix with complex entries and $\det(M_j) = a_j d_j - b_j c_j \neq 0$, define a Möbius transformation $\psi_j(z) = (a_j z + b_j)/(c_j z + d_j)$. It is well-known that

$$(\psi_1 \circ \psi_2 \circ \cdots \circ \psi_n)(z) = (A_n z + B_n)/(C_n z + D_n), \tag{5.3}$$

where

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = M_1 M_2 \cdots M_n. \tag{5.4}$$

If $\mathcal{B}$ is a finite set of complex numbers b such that $\mathrm{Re}(b) \geq \gamma > 0$ for all $b \in \mathcal{B}$, we define $\mathcal{B}_\nu$ as before by

$$\mathcal{B}_\nu = \{\omega = (b_1, b_2, \ldots, b_\nu) : b_j \in \mathcal{B} \text{ for } 1 \leq j \leq \nu\}$$

and $\theta_\omega = \theta_{b_n} \circ \theta_{b_{n-1}} \cdots \circ \theta_{b_1}$. Given $\omega = (b_1, b_2, \ldots, b_\nu) \in \mathcal{B}_\nu$, we define

$$\tilde{\omega} = (b_\nu, b_{\nu-1}, \ldots, b_1) \tag{5.5}$$

so

$$\theta_{\tilde{\omega}} = \theta_{b_1} \circ \theta_{b_2} \cdots \circ \theta_{b_n}. \tag{5.6}$$

For Λ_s as in (5.2) $\nu \geq 1$, and $f \in C^m_{\mathbb{C}}(\bar{H})$, recall that

$$(\Lambda_s^\nu(f))(z) = \sum_{\omega \in \mathcal{B}_\nu} \left| \frac{d\theta_\omega(z)}{dz} \right|^s f(\theta_\omega(z)) = \sum_{\omega \in \mathcal{B}_\nu} \left| \frac{d\theta_{\tilde{\omega}}(z)}{dz} \right|^s f(\theta_{\tilde{\omega}}(z)).$$

The following lemma allows us to apply Theorem 4.1 to Λ_s in (5.2).

Lemma 5.1. *Let b_1 and b_2 be complex numbers with $\mathrm{Re}(b_j) \geq \gamma \geq 1$ for $j = 1, 2$. If $\psi_j(z) = 1/(z + b_j)$ for $\mathrm{Re}(z) \geq 0$ and $\theta = \psi_1 \circ \psi_2$, then for all z, w with $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(w) \geq 0$,*

$$|\theta(z) - \theta(w)| \leq (\gamma^2 + 1)^{-2} |z - w|.$$

Proof. It suffices to prove that $|(d\theta/dz)(z)| \leq (\gamma^2 + 1)^{-2}$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geq 0$. Using (5.3) and (5.4) we see that

$$|(d\theta/dz)(z)| = |b_1|^{-2} |z + (1/b_1) + b_2|^{-2},$$

so it suffices to prove that $|b_1|^2 \, |z + (1/b_1) + b_2|^2 \geq (\gamma^2 + 1)^2$ for $\mathrm{Re}(z) \geq 0$. If we write $b_1 = u + iv$ with $u \geq \gamma$,

$$\mathrm{Re}(z + (1/b_1) + b_2) \geq u/(u^2 + v^2) + \gamma,$$

so

$$|z + (1/b_1) + b_2|^2 \geq [u/(u^2 + v^2) + \gamma]^2$$

and

$$|b_1|^2\,|z+(1/b_1)+b_2|^2 \geq (u^2+v^2)\Big[\frac{u^2}{(u^2+v^2)^2}+\frac{2u\gamma}{(u^2+v^2)}+\gamma^2\Big]$$
$$=\frac{u^2}{(u^2+v^2)}+2u\gamma+\gamma^2(u^2+v^2)=g(u,v).$$

Because $u \geq \gamma$, $g(u,0) = 1 + 2\gamma^2 + \gamma^4 = (\gamma^2+1)^2$. Using the fact that $u \geq \gamma \geq 1$, we also see that for $v \geq 0$

$$\frac{\partial g(u,v)}{\partial v} = \frac{-u^2(2v)}{(u^2+v^2)^2} + 2\gamma^2 v \geq 0,$$

which implies that $g(u,v) \geq g(u,0) = (\gamma^2+1)^2$ for $u \geq \gamma$ and $v \geq 0$. Since $g(u,-v) = g(u,v)$, $g(u,v) \geq (\gamma^2+1)^2$ for $v \leq 0$ and $u \geq \gamma$. □

With the aid of Lemma 5.1, the following theorem is an immediate corollary of Theorem 4.1.

Theorem 5.2. *Assume (H5.1) and let H be a bounded, open mildly regular subset of $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ such that $H \supset G_\gamma$, where G_γ is defined by (5.1). For a given positive integer m and for $s > 0$, let $X = C^m_{\mathbb{C}}(\bar{H})$ and $Y = C_{\mathbb{C}}(\bar{H})$ and let $\Lambda_s : X \to X$ and $L_s : Y \to Y$ be given by (5.2). If $r(\Lambda_s)$ (respectively, $r(L_s)$) denotes the spectral radius of Λ_s (respectively, L_s), we have $r(\Lambda_s) > 0$ and $r(\Lambda_s) = r(L_s)$. If $\rho(\Lambda_s)$ denotes the essential spectral radius of Λ_s,*

$$\rho(\Lambda_s) \leq (\gamma^2+1)^{-m} r(\Lambda_s).$$

For each $s > 0$, there exists $v_s \in X$ such that $v_s(z) > 0$ for all $z \in \bar{H}$ and $\Lambda_s(v_s) = r(\Lambda_s)v_s$. All the statements of Theorem 4.1 are true in this context whenever Λ and L in Theorem 4.1 are replaced by Λ_s and L_s respectively.

In the notation of Theorem 5.2, it follows from (4.9) that for any multi-index $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 + \alpha_2 \leq m$ and for $z = x + iy = (x, y)$

$$\lim_{\nu\to\infty} \frac{D^\alpha\left(\sum_{\omega\in\mathcal{B}_\nu}\left|\frac{d}{dz}\theta_\omega(z)\right|^s\right)}{\sum_{\omega\in\mathcal{B}_\nu}\left|\frac{d}{dz}\theta_\omega(z)\right|^s} = \frac{D^\alpha v_s(x,y)}{v_s(x,y)}, \tag{5.7}$$

where $D^\alpha = (\partial/\partial x)^{\alpha_1}(\partial/\partial y)^{\alpha_2}$ and the convergence is uniform in $(x, y) := z \in \bar{H}$.

Lemma 5.3. *Let b_j, $j \geq 1$ be a sequence of complex numbers with $\mathrm{Re}(b_j) \geq \gamma > 0$ for all j. For complex numbers z, define $\theta_{b_j}(z) = (z + b_j)^{-1}$ and define matrices $M_j = \left(\begin{smallmatrix} 0 & 1 \\ 1 & b_j \end{smallmatrix}\right)$. Then for $n \geq 1$,*

$$M_1 M_2 \cdots M_n = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}, \tag{5.8}$$

where $A_0 = 0$, $A_1 = 1$, $B_0 = 1$, $B_1 = b_1$ and for $n \geq 1$,

$$A_{n+1} = A_{n-1} + b_{n+1}A_n \text{ and } B_{n+1} = B_{n-1} + b_{n+1}B_n. \tag{5.9}$$

Also,

$$(\theta_{b_1} \circ \theta_{b_2} \cdots \circ \theta_{b_n})(z) = (A_{n-1}z + A_n)/(B_{n-1}z + B_n),$$

and we have

$$\mathrm{Re}(B_n/B_{n-1}) \geq \gamma \tag{5.10}$$

and

$$\Big|\frac{d}{dz}\Big[\frac{A_{n-1}z + A_n}{B_{n-1}z + B_n}\Big]\Big|^s = |B_{n-1}|^{-2s}|z + B_n/B_{n-1}|^{-2s}. \tag{5.11}$$

Proof. Equation (5.8) follows by induction on n. It is obviously true for $n = 1$. If we assume that (5.8) is satisfied for some $n \geq 1$, then

$$M_1M_2\cdots M_nM_{n+1} = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_{n+1} \end{pmatrix} = \begin{pmatrix} A_n & A_{n-1} + b_{n+1}A_n \\ B_n & B_{n-1} + b_{n+1}B_n \end{pmatrix},$$

which proves (5.8) with A_{n+1} and B_{n+1} defined by (5.9). Similarly, we prove (5.10) by induction on n. The case $n = 1$ is obvious, Assuming that (5.9) is satisfied for some $n \geq 1$, we obtain from (5.9) that

$$B_{n+1}/B_n = B_{n-1}/B_n + b_{n+1}.$$

Because $\mathrm{Re}(w) \geq \gamma$, where $w := B_n/B_{n-1}$, we see that $|1/w - 1/(2\gamma)| \leq 1/(2\gamma)$ and $\mathrm{Re}(1/w) = \mathrm{Re}(B_{n-1}/B_n) \geq 0$, so

$$\mathrm{Re}(B_{n+1}/B_n) \geq \mathrm{Re}(B_{n-1}/B_n) + \mathrm{Re}(b_{n+1}) \geq \gamma.$$

Hence (5.9) is satisfied for all $n \geq 1$. Because $\det(M_j) = -1$ for all $j \geq 1$, we get that $\det\left(\begin{smallmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{smallmatrix}\right) = (-1)^n$, and (5.11) follows. □

Before proceeding further, it will be convenient to establish some elementary calculus propositions. For $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and $s > 0$, define

$$G(u, v; s) = (u^2 + v^2)^{-s}.$$

Define $D_1 = (\partial/\partial u)$, so $D_1^m = (\partial/\partial u)^m$ for positive integers m; similarly, let $D_2 = (\partial/\partial v)$ and $D_2^m = (\partial/\partial v)^m$.

Lemma 5.4. *For positive integers m, there exist polynomials in u and v with coefficients depending on s, $P_m(u, v; s)$ and $Q_m(u, v; s)$, such that*

$$D_1^m G(u, v; s) = P_m(u, v; s)G(u, v; s + m),$$
$$D_2^m G(u, v; s) = Q_m(u, v; s)G(u, v; s + m).$$

Furthermore, we have $P_1(u, v; s) = -2su$, $Q_1(u, v; s) = -2sv$, and for positive integers m,

$$P_{m+1}(u, v; s) = (u^2 + v^2)(D_1P_m(u, v; s)) - 2(s + m)uP_m(u, v; s)$$

and

$$Q_{m+1}(u, v; s) = (u^2 + v^2)(D_2Q_m(u, v; s)) - 2(s + m)vQ_m(u, v; s).$$

Proof. If $m = 1$,

$$D_1G(u, v; s) = (-2su)\, G(u, v; s{+}1), \qquad D_2G(u, v; s) = (-2sv)(u^2{+}v^2; s{+}1),$$

so $P_1(u, v; s) = -2su$ and $Q_1(u, v; s) = -2sv$.

We now argue by induction and assume we have proved the existence of $P_j(u, v; s)$ and $Q_j(u, v; s)$ for $1 \leq j \leq m$. It follows that

$$\begin{aligned}D_1^{m+1}G(u,v;s) &= D_1[P_m(u,v;s)G(u,v;s+m)]\\ &= [D_1P_m(u,v;s)]G(u,v;s+m)]\\ &\quad + P_m(u,v;s)[-2(s+m)u]G(u,v;s+m+1)\\ &= [(u^2+v^2)(D_1P_m(u,v;s))\\ &\quad - 2(s+m)uP_m(u,v;s)]G(u,v;s+m+1).\end{aligned}$$

This proves the lemma with

$$P_{m+1}(u,v;s) := (u^2+v^2)(D_1P_m(u,v;s)) - 2(s+m)uP_m(u,v;s).$$

An exactly analogous argument, which we leave to the reader, shows that

$$Q_{m+1}(u,v;s) := (u^2+v^2)(D_2Q_m(u,v;s)) - 2(s+m)vQ_m(u,v;s). \qquad \square$$

An advantage of working with Möbius transformations is that one can easily obtain tractable formulas for expressions like $(\theta_{b_1}\circ\theta_{b_2}\cdots\circ\theta_{b_n})(z)$. Such formulas allow more precise estimates for the left hand side of (4.9) than we obtained in Section 5 of [13].

Lemma 5.5. *In the notation of Lemma 5.4, for all $(u,v)\in\mathbb{R}^2\setminus\{(0,0)\}$, for all $s>0$, and all positive integers m, $P_m(u,v;s)=Q_m(v,u;s)$.*

Proof. Fix $s>0$. We have $P_1(u,v;s)=Q_1(v,u;s)$ for all $(u,v)\neq(0,0)$. Arguing by mathematical induction, assume that for some positive integer m we have proved that $P_m(u,v;s)=Q_m(v,u;s)$ for all $(u,v)\neq(0,0)$. For a fixed $(u,v)\neq(0,0)$, we obtain, by virtue of the recursion formula in Lemma 5.4,

$$\begin{aligned}P_{m+1}(v,u;s) &= (u^2+v^2)\lim_{\Delta v\to 0}\frac{P_m(v+\Delta v,u;s)-P_m(v,u;s)}{\Delta v}\\ &\quad - 2(s+m)vP_m(v,u;s)\\ &= (u^2+v^2)\lim_{\Delta v\to 0}\frac{Q_m(u,v+\Delta v;s)-Q_m(u,v;s)}{\Delta v}\\ &\quad - 2(s+m)vQ_m(u,v;s)\\ &= Q_{m+1}(u,v;s).\end{aligned}$$

By mathematical induction, we conclude that $P_n(u,v;s)=Q_n(v,u;s)$ for all positive integers n. $\square$

Remark 5.1. By using the recursion formula in Lemma 5.4, one can easily compute $P_j(u,v;s)$ for $1\le j\le 4$.

$$\begin{aligned}&P_1(u,v;s) = -2su,\\ &P_2(u,v;s) = 2s(2s+1)u^2 - 2sv^2,\\ &P_3(u,v;s) = -2s(2s+1)(2s+2)u^3 + (2s)(2s+2)(3)uv^2,\\ &P_4(u,v;s) = (2s)(2s+2)[(2s+1)(2s+3)u^4 - 6(2s+3)u^2v^2 + 3v^4].\end{aligned}$$

By virtue of Lemma 5.5, we also obtain formulas for $Q_j(v,u;s) = P_j(u,v;s)$. Also, Lemmas 5.4 and 5.5 imply that

$$\frac{D_1^jG(u,v;s)}{G(u,v;s)} = \frac{P_j(u,v;s)}{(u^2+v^2)^j}, \qquad \frac{D_2^jG(u,v;s)}{G(u,v;s)} = \frac{P_j(v,u;s)}{(u^2+v^2)^j}$$

and the latter formulas will play a useful role in this section. In particular, for a given constant $\gamma > 0$, we shall need good estimates for

$$\sup\left\{\frac{D_k^j G(u,v;s)}{G(u,v;s)} : u \ge \gamma, v \in \mathbb{R}\right\} \text{ and } \inf\left\{\frac{D_k^j G(u,v;s)}{G(u,v;s)} : u \ge \gamma, v \in \mathbb{R}\right\}$$

where $k = 1, 2$ and $1 \le j \le 4$. Although the arguments used to prove these estimates are elementary, these results will play a crucial role in our later work.

Lemma 5.6. *Let $\gamma > 0$ be a given constant and assume that $u \ge \gamma$ and $v \in \mathbb{R}$. Let $D_1 = (\partial/\partial u)$ and $G(u,v;s) = (u^2+v^2)^{-s}$, where $s > 0$. For $j \ge 1$ we have*

$$\frac{D_1^j G(u,v;s)}{G(u,v;s)} = \frac{P_j(u,v;s)}{(u^2+v^2)^j},$$

where $P_j(u,v;s)$ is as defined in Remark 5.1; and the following estimates are satisfied.

$$-\frac{2s}{\gamma} \le \frac{D_1 G(u,v;s)}{G(u,v;s)} < 0,$$

$$-\frac{s}{4\gamma^2(s+1)} \le \frac{D_1^2 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)}{\gamma^2},$$

$$-\frac{2s(2s+1)(2s+2)}{\gamma^3} \le \frac{D_1^3 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+2)}{\gamma^3(s+2)^2},$$

$$-\frac{2s(s+1)(2s+2)(3)}{\gamma^4} \le \frac{D_1^4 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)(2s+2)(2s+3)}{\gamma^4}.$$

Proof. By Remark 5.1,

$$\frac{D_1^j G(u,v;s)}{G(u,v;s)} = \frac{P_j(u,v;s)}{(u^2+v^2)^j},$$

and Remark 5.1 provides formulas for $P_j(u,v;s)$. It follows that

$$\frac{D_1^j G(u,v;s)}{G(u,v;s)} = \frac{-2su}{u^2+v^2} < 0.$$

Since

$$\frac{2su}{u^2+v^2} \le \frac{2su}{u^2} \le \frac{2s}{\gamma},$$

we also see that

$$\frac{D_1 G(u,v;s)}{G(u,v;s)} \ge -\frac{2s}{\gamma}.$$

Using Remark 5.1, we see that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+1)u^2 - 2sv^2}{(u^2+v^2)^2},$$

so

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)u^2}{(u^2+v^2)^2}.$$

Since

$$\frac{u^2}{(u^2+v^2)^2} \le \frac{u^2}{u^4} \le \frac{1}{\gamma^2},$$

we find that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} \le \frac{2s(2s+1)}{\gamma^2},$$

If we write $v^2 = \rho u^2$, we see that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+1-\rho)}{u^2(1+\rho)^2},$$

and if $0 \le \rho \le 2s+1$, we obtain the upper bound given above and a lower bound of zero. If $\rho > 2s+1$, we see that

$$\frac{D_1^2 G(u,v;s)}{G(u,v;s)} \ge \frac{2s}{\gamma^2} \inf\left\{\frac{2s+1-\rho}{(1+\rho)^2} : \rho > 2s+1\right\}.$$

It is a simple calculus exercise to show that

$$\inf\left\{\frac{2s+1-\rho}{(1+\rho)^2} : \rho > 2s+1\right\} = -\frac{1}{8(s+1)},$$

achieved at $\rho = 4s+3$; and this gives the lower estimate $-s/[4\gamma^2(s+1)]$ of the lemma.

Using Remark 5.1 again, we see that

$$\frac{D_1^3 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+2)u[-(2s+1)u^2+3v^2]}{(u^2+v^2)^3}.$$

It follows that

$$\begin{aligned}\frac{D_1^3 G(u,v;s)}{G(u,v;s)} &\ge -2s(2s+1)(2s+2)\left[\frac{u}{(u^2+v^2)}\right]^3 \\ &\ge -2s(2s+1)(2s+2)\left[\frac{1}{u}\right]^3 \ge -2s(2s+1)(2s+2)\frac{1}{\gamma^3}.\end{aligned}$$

On the other hand, if we write $v^2 = \rho u^2$, then

$$\begin{aligned}\frac{D_1^3 G(u,v;s)}{G(u,v;s)} &= \frac{2s(2s+2)}{u^3}\frac{[3\rho-(2s+1)]}{(1+\rho)^3} \\ &\le \frac{2s(2s+2)}{\gamma^3}\sup\left\{\frac{3\rho-(2s+1)}{(1+\rho)^3} : \rho \ge 0\right\}.\end{aligned}$$

Once again, a straightforward calculus argument shows that

$$\sup\left\{\frac{3\rho-(2s+1)}{(1+\rho)^3} : \rho \ge 0\right\} = \frac{1}{(s+2)^2}$$

and the supremum is achieved at $\rho = s+1$. Using this fact, we obtain the upper estimate of the lemma.

Finally, we obtain from Remark 5.1 that

$$\frac{D_1^4 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+2)[(2s+1)(2s+3)u^4 - 6(2s+3)u^2v^2 + 3v^4]}{(u^2+v^2)^4}.$$

Dropping the negative term in the numerator and observing that

$$3 \le (2s+1)(2s+3) \quad \text{and} \quad u^4 + v^4 \le (u^2+v^2)^2,$$

we see that

$$\begin{aligned}\frac{D_1^4 G(u,v;s)}{G(u,v;s)} &\le \frac{(2s)(2s+1)(2s+2)(2s+3)(u^4+v^4)}{(u^2+v^2)^4}\\ &\le \frac{(2s)(2s+1)(2s+2)(2s+3)}{(u^2+v^2)^2} \le \frac{(2s)(2s+1)(2s+2)(2s+3)}{\gamma^4}.\end{aligned}$$

On the other hand, because $-u^4 - v^4 \le -2u^2v^2$, we obtain that

$$\begin{aligned}-\frac{D_1^4 G(u,v;s)}{G(u,v;s)} &\le \frac{(2s)(2s+2)[-3u^4 + 6(2s+3)u^2v^2 - 3v^4]}{(u^2+v^2)^4}\\ &\le \frac{3(2s)(2s+2)[-2u^2v^2 + (4s+6)u^2v^2]}{(u^2+v^2)^4}\\ &\le \frac{3(2s)(2s+2)[4(s+1)(u^2+v^2)^2/4]}{(u^2+v^2)^4}\\ &\le \frac{3(2s)(2s+2)(s+1)}{(u^2+v^2)^2} \le \frac{3(2s)(2s+2)(s+1)}{\gamma^4},\end{aligned}$$

which gives the lower estimate of Lemma 5.6. □

The following lemma gives analogous estimates for

$$\frac{D_2^j G(u,v;s)}{G(u,v;s)} = \frac{P_j(v,u;s)}{(u^2+v^2)^j}.$$

Lemma 5.7. *Let $\gamma > 0$ be a given real number, $D_2 = (\partial/\partial v)$ and for $s > 0$ and $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$, define $G(u,v;s) = (u^2+v^2)^{-s}$, If $u \ge \gamma$ and $v \in \mathbb{R}$, we have the following estimates.*

$$\begin{aligned}\frac{|D_2 G(u,v;s)|}{G(u,v;s)} &\le \frac{s}{\gamma},\\ -\frac{2s}{\gamma^2} \le \frac{D_2^2 G(u,v;s)}{G(u,v;s)} &\le \frac{2s(2s+1)}{4\gamma^2},\\ \frac{|D_2^3 G(u,v;s)|}{G(u,v;s)} &\le \frac{2s(2s+2)}{\gamma^3} \max\left\{\frac{25\sqrt{5}}{72}, \frac{2s+1}{8}\right\}\\ -\frac{2s(s+1)(2s+2)(3)}{\gamma^4} \le \frac{D_2^4 G(u,v;s)}{G(u,v;s)} &\le \frac{2s(2s+1)(2s+2)(2s+3)}{\gamma^4}.\end{aligned}$$

Proof. By Remark 5.1, $P_1(v,u;s) = -2sv$, so

$$\frac{|D_2 G(u,v;s)|}{G(u,v;s)} = \frac{2s|v|}{u^2+v^2}.$$

The map $w \mapsto w/(u^2+w^2)$ has its maximum on $[0,\infty)$ at $w = u$, so $(2s|v|/(u^2+v^2) \le s/u \le s/\gamma$; and we obtain the first inequality in Lemma 5.7. Using Remark 5.1 again, we see that

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} = \frac{2s[(2s+1)v^2 - u^2]}{(u^2+v^2)^2}.$$

It follows that

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} = 2s(2s+1)\frac{|v|^2}{(u^2+v^2)^2}.$$

The map $v \mapsto |v|/(u^2+v^2)$ has its maximum at $|v| = u$, so $[|v|/(u^2+v^2)]^2 \le 1/(4u^2) \le 1/(4\gamma^2)$, and

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} = \frac{2s(2s+1)}{4\gamma^2}.$$

Similarly, one obtains

$$\frac{D_2^2 G(u,v;s)}{G(u,v;s)} \ge -\frac{2su^2}{(u^2+v^2)^2} \ge -\frac{2s}{u^2} \ge -\frac{2s}{\gamma^2}.$$

With the aid of Remark 5.1 again, we see that

$$\frac{D_2^3 G(u,v;s)}{G(u,v;s)} = 2s(2s+2)v\frac{[-(2s+1)v^2+3u^2]}{(u^2+v^2)^3} := A(u,v).$$

For a fixed $u \ge \gamma$, $v \mapsto A(u,v)$ is an odd function of v, so if $\alpha(u) = \sup\{A(u,v) : v \in \mathbb{R}\}$, $-\alpha(u) = \inf\{A(u,v) : v \in \mathbb{R}\}$. If $v \le 0$,

$$\begin{aligned} A(u,v) &\le (2s)(2s+1)(2s+2)\left[\frac{|v|}{u^2+v^2}\right]^3 \le (2s)(2s+1)(2s+2)\left[\frac{u}{2u^2}\right]^3 \\ &\le \frac{(2s)(2s+1)(2s+2)}{8\gamma^3}. \end{aligned}$$

If $v > 0$,

$$A(u,v) \le (2s)(2s+2)(3u^2)\frac{v}{(u^2+v^2)^3}.$$

A calculation shows that $v \mapsto v/(u^2+v^2)^3$ achieves its maximum for $v \ge 0$ at $v = u/\sqrt{5}$, so for $v > 0$,

$$A(u,v) \le (2s)(2s+2)(3u^{-3})[\sqrt{5}(6/5)^3]^{-1} \le (2s)(2s+2)\gamma^{-3}(25\sqrt{5}/72).$$

Note that $25\sqrt{5}/72 \approx .7764 < 1$. Using Remark 5.1 again, we see that

$$\frac{D_2^4 G(u,v;s)}{G(u,v;s)} = 2s(2s+2)\frac{[(2s+1)(2s+3)v^4 - 6(2s+3)u^2v^2 + 3u^4]}{(u^2+v^2)^4}.$$

Since $u^4 + v^4 \le (u^2+v^2)^2$, it follows easily that

$$\begin{aligned} \frac{D_2^4 G(u,v;s)}{G(u,v;s)} &\le 2s(2s+2)(2s+1)(2s+3)\frac{u^4+v^4}{(u^2+v^2)^4} \\ &\le 2s(2s+2)(2s+1)(2s+3)\gamma^{-4}. \end{aligned}$$

Similarly, we see that

$$\begin{aligned} (2s+1)(2s+3)v^4 - 6(2s+3)u^2v^2 + 3u^4 &\ge 3(u^4+v^4) - 6(2s+3)[(u^2+v^2)/2]^2 \\ &\ge 3(u^2+v^2)^2 - 6[(u^2+v^2)/2]^2 - 6(2s+3)[(u^2+v^2)/2]^2. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{D_2^4 G(u,v;s)}{G(u,v;s)} &\ge 2s(2s+2)\frac{3 - 3/2 - 3/2(2s+3)}{(u^2+v^2)^2} \\ &\ge -(2s)(2s+2)3(s+1)(u^2+v^2)^{-2} \ge -(2s)(2s+2)(3s+3)\gamma^{-4}, \end{aligned}$$

which completes the proof of Lemma 5.7. Note that $(2s)(2s+1)(2s+2)(2s+3) \geq 2s(2s+2)(3s+3)$. □

Remark 5.2. Lemmas 5.6 and 5.7 show that whenever $u \geq \gamma > 0$, $s > 0$, $k = 1$ or $k = 2$, and $1 \leq j \leq 4$,

$$\frac{|D_k^j G(u,v;s)|}{G(u,v;s)} \leq (2s)(2s+1)\cdots(2s+j-1)\gamma^{-j}.$$

We have not determined whether the above inequality holds for all $j \geq 1$.

Using Lemmas 5.6 and 5.7, we can give uniform estimates for the quantities $(\partial/\partial x)^j v_s(x,y)/v_s(x,y)$ and $(\partial/\partial y)^j v_s(x,y)/v_s(x,y)$, where $s > 0$, $1 \leq j \leq 4$, and $v_s(x,y)$ is the unique (to within normalization) strictly positive eigenfunction of the linear operator $\Lambda_s : C^m_{\mathbb{C}}(\bar{H}) \to C^m_{\mathbb{C}}(\bar{H})$ in (5.2) for $m \geq 1$.

Theorem 5.8. *Let s denote a positive real and let $\mathcal{B}$ and θ_b, $b \in \mathcal{B}$, be as in (H5.1). Let H be a bounded, mildly regular open subset of $\mathbb{C} := \mathbb{R}^2$ such that $H \supset G_\gamma = \{z \in \mathbb{C} : |z - 1/(2\gamma)| < 1/(2\gamma)\}$, and $\mathrm{Re}(z) > 0$ for all $z \in H$, so $\theta_b(H) \subset G_\gamma$ for all $b \in \mathcal{B}$. For a positive integer m, define a complex Banach space $C^m_{\mathbb{C}}(\bar{H}) = X$ and let $\Lambda_s : X \to X$ be defined as in (5.2). Then Λ_s has a unique (to within normalization) strictly positive eigenfunction $v_s \in X$ and $v_s \in C^\infty$. Furthermore, we have the following estimates for $(x,y) \in \bar{H}$.*

$$-\frac{2s}{\gamma} \leq \frac{\partial v_s(x,y)}{\partial x}[v_s(x,y)]^{-1} \leq 0, \tag{5.12}$$

$$-\frac{s}{4\gamma^2(s+1)} \leq \frac{\partial^2 v_s(x,y)}{\partial x^2}[v_s(x,y)]^{-1} \leq \frac{2s(2s+1)}{\gamma^2}, \tag{5.13}$$

$$-\frac{2s(2s+1)(2s+2)}{\gamma^3} \leq \frac{\partial^3 v_s(x,y)}{\partial x^3}[v_s(x,y)]^{-1} \leq \frac{(2s)(2s+2)}{\gamma^3(s+2)^2}, \tag{5.14}$$

$$\begin{aligned} &-\frac{2s(2s+2)(3s+3)}{\gamma^4} \\ &\leq \frac{\partial^4 v_s(x,y)}{\partial x^4}[v_s(x,y)]^{-1} \\ &\leq \frac{(2s)(2s+1)(2s+2)(2s+3)}{\gamma^4}, \end{aligned} \tag{5.15}$$

$$\left|\frac{\partial v_s(x,y)}{\partial y}\right|[v_s(x,y)]^{-1} \leq \frac{s}{\gamma}, \tag{5.16}$$

$$-\frac{2s}{\gamma^2} \leq \frac{\partial^2 v_s(x,y)}{\partial y^2}[v_s(x,y)]^{-1} \leq \frac{2s(2s+1)}{4\gamma^2}, \tag{5.17}$$

$$\left|\frac{\partial^3 v_s(x,y)}{\partial y^3}\right|[v_s(x,y)]^{-1} \leq \frac{(2s)(2s+2)}{\gamma^3}\max\{25\sqrt{5}/72, (2s+1)/8\}, \tag{5.18}$$

$$\begin{aligned} -\frac{2s(2s+2)(3s+3)}{\gamma^4} &\leq \frac{\partial^4 v_s(x,y)}{\partial y^4}[v_s(x,y)]^{-1} \\ &\leq \frac{(2s)(2s+1)(2s+2)(2s+3)}{\gamma^4}. \end{aligned} \tag{5.19}$$

Hence, if $D_1 = \partial/\partial x$ *and* $D_2 = \partial/\partial y$*, we have for* $k = 1, 2$ *and* $1 \le j \le 4$ *that*

$$\frac{|D_k^j v_s(x,y)|}{v_s(x,y)} \le \frac{(2s)(2s+1)\cdots(2s+j-1)}{\gamma^j}. \tag{5.20}$$

Proof. For any integer $m \ge 1$, we can view Λ_s as a bounded linear operator of $C^m_{\mathbb{C}}(\bar{H})$ to $C^m_{\mathbb{C}}(\bar{H})$. We know that Λ_s has a strictly positive eigenfunction $v_s(x,y) \in C^m_{\mathbb{C}}(\bar{H})$ such that $\sup\{v_s(x,y) : (x,y) \in \bar{H}\} = 1$. By the uniqueness of this eigenfunction, $v_s(x,y)$ must actually be C^∞.

Using the notation of (5.5) and (5.6) and also using (5.11) in Lemma 5.3, we see that

$$\left|\frac{d}{dz}\theta_{\tilde{\omega}}(z)\right|^s = |B_{n-1}|^{-2s}|z + B_n/B_{n-1}|^{-2s}.$$

By Lemma 5.3, $\mathrm{Re}(B_n/B_{n-1}) \ge \gamma_\omega \ge \gamma$, so writing $\mathrm{Im}(B_n/B_{n-1}) = \delta_\omega$, we obtain that for $k = 1, 2$ and $1 \le j$,

$$\begin{aligned} &D_k^j\left(\left|\frac{d}{dz}\theta_{\tilde{\omega}}(z)\right|^s\right)\left|\frac{d}{dz}\theta_{\tilde{\omega}}(z)\right|^s \\ &\quad = \left(D_k^j\left[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\right]^{-s}\right)\left[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\right]^s. \end{aligned} \tag{5.21}$$

However, if we write $(x + \gamma_\omega) = u \ge \gamma$ and $(y + \delta_\omega) = v$, we see that

$$\begin{aligned} &\left(\left(\frac{\partial}{\partial x}\right)^j\left[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\right]^{-s}\right)\left[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\right]^{-s} \\ &\quad = \left[\left(\frac{\partial}{\partial u}\right)^j G(u,v;s)\right][G(u,v;s]^{-1}, \end{aligned} \tag{5.22}$$

where the right hand side of the above equation is evaluated at $u = x + \gamma_\omega$ and $v = y + \delta_\omega$. If we combine (5.21) and (5.22) with the estimates in Lemma 5.6 and if we then use (5.7), we obtain the estimates on $(\partial/\partial x)^j v_s(x,y)$ given in (5.12) - (5.15).

Similarly, we have

$$\begin{aligned} &\left(\left(\frac{\partial}{\partial y}\right)^j\left[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\right]^{-s}\right)\left[(x+\gamma_\omega)^2 + (y+\delta_\omega)^2\right]^{-s} \\ &\quad = \left[\left(\frac{\partial}{\partial v}\right)^j G(u,v;s)\right][G(u,v;s]^{-1}. \end{aligned} \tag{5.23}$$

If we combine (5.21) and (5.23) with the estimates in Lemma 5.7 and if we then use (5.7), we obtain the estimates on $(\partial/\partial y)^j v_s(x,y)$ given in (5.16) - (5.19). □

Remark 5.3. Let $H, \mathcal{B}$, and θ_b, $b \in \mathcal{B}$, be as in Theorem 5.8 and let R and α be positive reals such that $R \ge \sup\{|b|, b \in \mathcal{B}\}$. Define $\theta_0 : \bar{H} \to \bar{H}$ by $\theta_0(z) = 0$ for all $z \in \bar{H}$ and let $L_{s,R,\alpha} : X := C^m(\bar{H}) \to C^m(\bar{H})$ be as in (3.2) in Sect. 3. Notice that $L_{s,R,\alpha}$ satisfies all the hypotheses of Theorem 4.1, so all the conclusions of Theorem 4.1 hold. In particular, $L_{s,R,\alpha}$ has a unique (to within normalization) strictly positive eigenfunction $w_s \in C^m(\bar{H})$. Because the eigenfunction w_s is unique and $m \ge 1$ is arbitrary, $w_s \in C^m(\bar{H})$ for all $m \ge 1$.

We claim that exactly the same estimates given for v_s in Theorem 5.8 (i.e., (5.12) – (5.20)) also hold for w_s. To see this, define an index set $\mathcal{D} = \mathcal{B} \cup \{0\}$ and for $z \in \bar{H}$, define $g_\delta(z) = 1/|z+b|^{2s}$ if $\delta = b \in \mathcal{B}$ and $g_\delta(z) = \alpha$ if $\delta = 0$. As usual, if μ is a positive integer, let

$$\mathcal{D}_\mu = \{\omega = (\delta_1, \delta_2, \ldots, \delta_\mu) : \delta_k \in \mathcal{D} \text{ for } 1 \le k \le \mu\}.$$

Recall that for $\omega = (\delta_1, \delta_2, \ldots, \delta_\mu) \in \mathcal{D}_\mu$ and $\tilde{\omega}$ as in (5.5), our convention is that $\theta_{\tilde{\omega}} = \theta_{\delta_1} \circ \theta_{\delta_2} \circ \cdots \circ \theta_{\delta_\mu}$ and

$$g_{\tilde{\omega}}(z) = g_{\delta_\mu}(\theta_{\delta_{\mu-1}} \circ \theta_{\delta_{\mu-2}} \circ \cdots \circ \theta_{\delta_1}(z)) g_{\delta_{\mu-1}}(\theta_{\delta_{\mu-2}} \circ \theta_{\delta_{\mu-3}} \circ \cdots \circ \theta_{\delta_1}(z)) \\ \cdots g_{\delta_2}(\theta_{\delta_1}(z)) g_{\delta_1}(z).$$

If $D_1 = \partial/\partial x$ and $D_2 = \partial/\partial y$, for $k \ge 1$, $p = 1$ or 2, and $z = x + iy := (x, y)$, we know that

$$\frac{D_p^k w_s(x,y)}{w_s(x,y)} = \lim_{\mu \to \infty} \frac{D_p^k \Big(\sum_{\omega \in \mathcal{D}_\mu} g_{\tilde{\omega}}(x,y) \Big)}{\sum_{\omega \in \mathcal{D}_\mu} g_{\tilde{\omega}}(x,y)}.$$

If $\omega = (\delta_1, \delta_2, \ldots, \delta_\mu) \in \mathcal{D}_\mu$ and $\delta_k \ne 0$ for $1 \le k \le \mu$, we have seen in Lemmas 5.6 and 5.7 that $D_p^k g_{\tilde{\omega}}(x,y)/g_{\tilde{\omega}}(x,y)$ satisfies the same estimates given for $D_p^k v_s(x,y)/v_s(x,y)$ in equations (5.12)- (5.23). Thus assume that $\delta_t = 0$ for some t, $1 \le t \le \mu$ and $\delta_{t'} \ne 0$ for $1 \le t' < t$. A little thought shows that if $t = 1$, $g_{\tilde{\omega}}(z)$ is a positive constant. If $t = 2$, $g_{\tilde{\omega}}(z) = c(\omega) g_{\delta_1}(z)$, where $c(\omega)$ is a positive constant. Generally, if $2 \le t \le \mu$, $g_{\tilde{\omega}}(z) = c(\omega) g_{\tilde{\omega}_{t-1}}(z)$, where $c(\omega)$ is a positive constant and $\omega_{t-1} = (\delta_1, \delta_2, \ldots, \delta_{t-1}) \in \mathcal{D}_{t-1}$ and $\delta_1, \delta_2, \ldots, \delta_{t-1} \in \mathcal{B}$. It follows that $D_p^k g_{\tilde{\omega}}(x,y)/g_{\tilde{\omega}}(x,y) = 0$ if $t = 1$ and otherwise

$$D_p^k g_{\tilde{\omega}}(x,y)/g_{\tilde{\omega}}(x,y) = D_p^k g_{\tilde{\omega}_{t-1}}(x,y)/g_{\tilde{\omega}_{t-1}}(x,y).$$

By using Lemmas 5.6 and 5.7 again, it follows that if $\delta_t = 0$ for some t, $1 \le t \le \mu$, $D_p^k g_{\tilde{\omega}}(x,y)/g_{\tilde{\omega}}(x,y)$ is identically zero or satisfies the same estimates given for v_s in Theorem 5.8. Thus we see that $D_p^k w_s(x,y)/w_s(x,y)$ satisfies the same estimates given for $D_p^k v_s(x,y)/v_s(x,y)$ in Theorem 5.8.

Corollary 5.9. *Let notation and hypotheses be as in Remark 5.3. Then w_s satisfies inequalities (3.3)–(3.7) in Sect. 3. If $\mathcal{B}$ and H are symmetric under conjugation, $w_s(\bar{z}) = w_s(z)$ for all $z \in \bar{H}$.*

Proof. Let $H_1 \supset H$ be a convex, bounded open set such that $\mathrm{Re}(z) > 0$ for all $z \in H_1$. For $z \in \bar{H}_1$ and $L_{s,R,\alpha}$ given by (3.2), we can also view $L_{s,R,\alpha}$ as a bounded linear operator from $C^m_{\mathbb{C}}(\bar{H}_1) \to C^m_{\mathbb{C}}(\bar{H}_1)$, and this bounded linear operator has a unique strictly positive normalized eigenfunction $\hat{w}_s \in C^m_{\mathbb{C}}(\bar{H}_1)$. Uniqueness implies that $\hat{w}_s(z) = w_s(z)$ for all $z \in \bar{H}$. Thus, after replacing H by H_1, we can assume that H is convex.

If (x_1, y) and $(x_2, y) \in \bar{H}$ and $x_1 < x_2$, we obtain from (5.12) that

$$-\frac{2s}{\gamma}(x_2 - x_1) \le \int_{x_1}^{x_2} \frac{\partial}{\partial x} \log w_s(x,y)\, dx = \log\Big(\frac{w_s(x_2,y)}{w_s(x_1,y)}\Big) \le 0,$$

which gives (3.4). If (x_1, y) and $(x_2, y) \in \bar{H}$ and $y_1 < y_2$, we obtain from (5.16) that

$$-\frac{s}{\gamma}(y_2 - y_1) \le \int_{y_1}^{y_2} \frac{\partial}{\partial y} \log w_s(x, y)\, dy \le \frac{s}{\gamma}(y_2 - y_1),$$

which gives (3.5). For z_0 and $z_1 \in H$, define $z_t = (1-t)z_0 + tz_1$ and note that

$$\Big| \int_0^1 \frac{d}{dt} \log(w_s(z_t))\, dt \Big| = \Big| \log\Big(\frac{w_s(z_1)}{w_s(z_0)}\Big)\Big| \le \int_0^1 \Big| \frac{D_1 w_s(z_t)}{w_s(z_t)}(x_1 - x_0) + \frac{D_2 w_s(z_t)}{w_s(z_t)}(y_1 - y_0)\Big|\, dt,$$

where $z_j = (x_j, y_j)$, $j = 0, 1$. Using (5.12) and (5.16), we obtain

$$\Big| \log\Big(\frac{w_s(z_1)}{w_s(z_0)}\Big)\Big| \le \int_0^1 \Big| \frac{2s}{\gamma}|x_1 - x_0| + \frac{s}{\gamma}|y_1 - y_0|\Big|\, dt \le \frac{\sqrt{5}s}{\gamma}\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

which shows that w_s satisfies (3.3). Combining Remark 5.3 and Corollary 5.9, we see that w_s in Corollary 5.9 satisfies (3.3)–(3.7). It remains to verify the final statement in Corollary 5.9. If $\lambda_s = r(L_{s,R,\alpha}) > 0$, we know that w_s is the unique normalized, strictly positive eigenfunction of $L_{s,R,\alpha}$ with eigenvalue λ_s. Hence,

$$\lambda_s w_s(\bar{z}) = \sum_{b \in \mathcal{B}} \frac{1}{|\bar{z} + b|^{2s}} w_s(1/(\bar{z} + b)) + \alpha w_s(0) = \sum_{b \in \mathcal{B}} \frac{1}{|\bar{z} + \bar{b}|^{2s}} w_s(1/(\bar{z} + \bar{b})) + \alpha w_s(0).$$

If we define $\tilde{w}_s(z) = w_s(\bar{z})$ for all $z \in \bar{H}$, the above calculation shows

$$\lambda_s \tilde{w}_s(z) = \sum_{b \in \mathcal{B}} \frac{1}{|z + b|^{2s}} \tilde{w}_s(\theta_b(z)) + \alpha \tilde{w}_s(0) = \sum_{b \in \mathcal{B}} \frac{1}{|z + b|^{2s}} \tilde{w}_s(\theta_b(z)) + \alpha \tilde{w}_s(0).$$

By uniqueness of the strictly positive normalized eigenfunction, this implies that $\tilde{w}_s = w_s$, so $w_s(z) = w_s(\bar{z})$ for all $z \in H$. □

It remains to consider the case that $\mathcal{B}$ in Theorem 5.8 is countably infinite and that $s > 0$ is such that $\sum_{b \in \mathcal{B}}(1/|b|^{2s}) < \infty$.

Theorem 5.10. *Let $\mathcal{B}$ be a countably infinite set such that $\mathcal{B} \subseteq \{z \in \mathbb{C} : \mathrm{Re}(z) \ge \gamma \ge 1\}$. Assume that $s > 0$ is such that $\sum_{b \in \mathcal{B}}(1/|b|^{2s}) < \infty$. Let H and G_γ be as in Theorem 5.8. As was noted in Sect. 3 (see also Section 5 in [40,44]), $L_s : C_{\mathbb{C}}(\bar{H}) \to C_{\mathbb{C}}(\bar{H})$ defines a bounded linear map, where L_s is defined by (3.1), and L_s has a unique (to within scalar multiples) strictly positive Lipschitz eigenfunction v_s which satisfies inequalities (3.3)–(3.5) on $\bar{H}$. If $\mathcal{B}$ and H are symmetric under conjugation, $v_s(\bar{z}) = v_s(z)$ for all $z \in \bar{H}$.*

Proof. Select $R_0 > 0$ such that $\mathcal{B}_{R_0}$ is nonempty, and for $R \geq R_0$ define $L_{s,R}$ by

$$L_{s,R} = \sum_{b \in \mathcal{B}_R} \frac{f(\theta_b(z))}{|z+b|^{2s}}.$$

By Theorem 5.8, $L_{s,R}$ has a strictly positive C^∞ eigenfunction $v_{s,R}$ which satisfies (3.3)– (3.7) and has sup norm one. If d denotes the diameter of H, (3.3) implies that for all $z \in H$,

$$v_{s,R}(z) \geq \exp[-(\sqrt{5}s/\gamma)d]. \tag{5.24}$$

Now (3.3) implies that $z \mapsto \log(v_s(z))$ is Lipschitz with Lipschitz constant $\sqrt{5}s/\gamma$, which is independent of R. Using (5.24), it then follows that $z \mapsto v_s(z)$ is Lipschitz on H with Lipschitz constant C independent of $R \geq R_0$. By the Ascoli-Arzela theorem, there exists an increasing sequence of positive reals $R_j \to \infty$ such that $v_{s,R_j}(\cdot)$ converges uniformly on $\bar{H}$ to a function v_s. By uniform convergence, the function v_s satisfies (5.24) on $\bar{H}$, is strictly positive on $\bar{H}$, is continuous, and satisfies (3.3)–(3.5). If we define $\lambda_{s,R} = r(L_{s,R})$ for $R \geq R_0$, Lemma 2.3 implies that $\lambda_{s,R} \leq \lambda_{s,R'}$ whenever $R \leq R'$. If we define M_R by

$$M_R = \|L_{s,R}\| = \sup\Big\{\sum_{b \in \mathcal{B}_R} \frac{1}{|z+b|^{2s}} : z \in \bar{H}\Big\},$$

$r(L_{s,R}) \leq M_R$ and $M_R \leq M$, where

$$M = \sup\Big\{\sum_{b \in \mathcal{B}} \frac{1}{|z+b|^{2s}} : z \in \bar{H}\Big\}.$$

Using our assumption that $\sum_{b \in \mathcal{B}}(1/|b|^{2s}) < \infty$, one can prove that

$$\sum_{b \in \mathcal{B}} (1/|z+b|^{2s}) < \infty, \quad z \in \bar{H}$$

and that $\sum_{b \in \mathcal{B}_{R_j}}(1/|z+b|^{2s})$ converges uniformly on $\bar{H}$ to $\sum_{b \in \mathcal{B}}(1/|z+b|^{2s})$ as $j \to \infty$, so $z \mapsto \sum_{b \in \mathcal{B}}(1/|z+b|^{2s})$ is continuous and bounded on $\bar{H}$ and $M < \infty$. Since λ_{s,R_j} is an increasing sequence which is bounded by M, $\lambda_{s,R_j} \to \lambda_s > 0$. Using this information one can see that

$$\sum_{b \in \mathcal{B}_{R_j}} \big[v_{s,R_j}(\theta_b(z))/|z+b|^{2s}\big]$$

converges uniformly on $\bar{H}$ to $\sum_{b \in \mathcal{B}} \big[v_s(\theta_b(z))/|z+b|^{2s}\big] = \lambda_s v_s(z)$. Details are left to the reader.

Because v_s is a strictly positive eigenfunction on $\bar{H}$ for L_s with eigenvalue λ_s, Lemma 2.2 implies that $\lambda_s = r(L_s)$. Theorem 5.3 in [40] implies that L_s has no complex eigenvalues $\lambda \neq r(L_s)$ with $|\lambda| = r(L_s)$. If $\mathcal{B}$ and H are symmetric under conjugation, it was proved in Corollary 5.9 that $v_{s,R_j}(\bar{z}) = v_{s,R_j}(z)$ for all $z \in H$. The corresponding result for v_s follows by letting $R_j \to \infty$. □

The operator L_s induces a corresponding operator $\Lambda_s : C^{0,1}(\bar{H}) \to C^{0,1}(\bar{H})$, where $C^{0,1}(\bar{H})$ denotes the Banach space of Lipschitz continuous maps $f : \bar{H} \to \mathbb{C}$. One finds (e.g., see [37,40,46]) that $r(\Lambda_s) = r(L_s) := r > 0$ and there exists $r' < r$ such that $|\zeta| \le r'$ for all $\zeta \in \sigma(\Lambda_s)$, $\zeta \ne r(\Lambda_s)$. However, $r(L_s)$ may fail to be an isolated point in the spectrum of $L_s : C(\bar{H}) \to C(\bar{H})$, even for simple examples.

Theorem 5.11. *Let hypotheses and notation be as in Theorem 5.10. For a given number $R > 2$ and for $\mathcal{B}'_R := \{b \in \mathcal{B} : |b| > R\}$, assume that there exist $\delta_{s,R} > 0$ and $\eta_{s,R} \ge 0$ such that*

$$\eta_{s,R} v_s(0) \le \sum_{b \in \mathcal{B}'_R} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z)) \le \delta_{s,R} v_s(0).$$

Let $L_{s,R,\alpha}$ be defined by (3.2) and define $L_{s,R+} = L_{s,R,\alpha}$ for $\alpha = \delta_{s,R}$ and $L_{s,R-} = L_{s,R,\alpha}$ for $\alpha = \eta_{s,R}$. Then we have

$$r(L_{s,R-}) \le r(L_s) \le r(L_{s,R+}). \tag{5.25}$$

Proof. By our assumptions, if $\lambda_s := r(L_s)$,

$$L_s v_s = \lambda_s v_s \le L_{s,R+} v_s \qquad \text{and} \qquad L_{s,R-} v_s \le \lambda_s v_s.$$

Since v_s is strictly positive on $\bar{H}$, Lemma 2.2 implies (5.25). □

Now that we know the strictly positive eigenfunction v_s satisfies (3.3)–(3.5), when $\mathcal{B}$ is countably infinite, we can give estimates for the quantities $\delta_{s,R}$ and $\eta_{s,R}$ in Sect. 3.

Theorem 5.12. *Assume that $\mathcal{B} = I_1$ or $\mathcal{B} = I_2$ and let v_s be the unique strictly positive eigenfunction of L_s in (3.1), where we take $\bar{U} \supset D$ such that $0 \le x \le 1$ and $|y| \le 1/2$ for all $(x, y) \in \bar{U}$. Assume that $s > 1$ and $R > 2$. Then we have the following estimates:*

$$\sum_{b \in I_1, |b|>R} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z)) \le \exp\left(\frac{s}{\sqrt{R^2 - R}}\right)\left(\frac{R}{R-1}\right)^s \cdot \left[\left(\frac{1}{2s-1}\right)\left(\frac{1}{R-1}\right)^{2s-1} + \left(\frac{\pi}{2}\right)\left(\frac{1}{s-1}\right)\left(\frac{1}{R-\sqrt{2}}\right)^{2s-2}\right] v_s(0).$$

$$\sum_{b \in I_2, |b|>R} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z)) \le \exp\left(\frac{s}{\sqrt{R^2 - R}}\right)\left(\frac{R}{R-1}\right)^s \cdot \left[\left(\frac{\pi}{4}\right)\left(\frac{1}{s-1}\right)\left(\frac{1}{R-\sqrt{2}}\right)^{2s-2}\right] v_s(0).$$

Proof. First assume $\mathcal{B} = I_1$ in (3.1). Using (3.4) and (3.5), we have

$$v_s(\theta_b(z)) \le \exp(s|\theta_b(z)|) v_s(0).$$

Now for $z = x + iy \in D_h$ and $b = m + in \in I_1$, we have

$$\min_{(x,y)\in D_h} (x+m)^2 + (y+n)^2 \ge \min_{0 \le x \le 1} (x+m)^2 + \min_{|y| \le 1/2} (y+n)^2$$

$$\geq m^2 + (|n| - 1/2)^2 \geq m^2 + n^2 - |n|.$$

Hence, for $z \in D_h$,

$$\frac{1}{|z+b|^2} = \frac{1}{(x+m)^2 + (y+n)^2} \leq \frac{1}{m^2 + n^2 - |n|}.$$

Also, it is easy to check that if $m^2 + n^2 \geq R^2 > 1$,

$$\frac{1}{m^2 + n^2 - |n|} \leq \frac{R}{R-1} \frac{1}{m^2 + n^2} \leq \frac{1}{R^2 - R}.$$

Hence, for $m^2 + n^2 \geq R^2 > 1$ and $z \in D_h$,

$$\exp(s|\theta_b(z)|) \leq \exp\left(\frac{s}{\sqrt{m^2 + n^2 - |n|}}\right) \leq \exp\left(\frac{s}{\sqrt{R^2 - R}}\right).$$

It follows that

$$\sum_{b \in I_1, |b| > R} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z))$$
$$\leq \exp\left(\frac{s}{\sqrt{R^2 - R}}\right)\left(\frac{R}{R-1}\right)^s \left(\sum_{b \in I_1, |b| > R} \left(\frac{1}{m^2 + n^2}\right)^s\right) v_s(0).$$

Now for $n = 0$ and $m \geq R$,

$$\sum_{m \geq R} \frac{1}{m^{2s}} \leq \int_{R-1}^{\infty} \frac{1}{r^{2s}}\, dr = \frac{1}{2s-1}\left(\frac{1}{R-1}\right)^{2s-1}.$$

For $b = m + \mathrm{i}n \in I_1$ with $m \geq 1$, $n \geq 1$, and $|b| \geq R$, let

$$B(m,n) = \{(\xi, \eta) : m \leq \xi \leq m+1, n \leq \eta \leq n+1\}.$$

Then for $(u,v) \in B(m,n)$,

$$\frac{1}{(u-1)^2 + (v-1)^2} \geq \frac{1}{m^2 + n^2}.$$

Also,

$$\begin{aligned}(u-1)^2 + (v-1)^2 &\geq (m-1)^2 + (n-1)^2 = m^2 + n^2 - 2(m+n) + 2 \\ &\geq m^2 + n^2 - 2\sqrt{2}\sqrt{m^2 + n^2} + 2 \\ &= (\sqrt{m^2 + n^2} - \sqrt{2})^2 \geq (R - \sqrt{2})^2 \equiv R_1^2.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\substack{m \geq 1, n \geq 1 \\ m^2 + n^2 > R^2}} \left(\frac{1}{m^2 + n^2}\right)^s &\leq \sum_{\substack{m \geq 1, n \geq 1 \\ m^2 + n^2 > R^2}} \iint_{B(m,n)} \left(\frac{1}{(u-1)^2 + (v-1)^2}\right)^s du\, dv \\ &\leq \iint_{\substack{u \geq 0, v \geq 0 \\ u^2 + v^2 \geq R_1^2}} \left(\frac{1}{u^2 + v^2}\right)^s du\, dv \\ &= \frac{\pi}{2} \int_{R_1}^{\infty} \frac{1}{r^{2s}} r\, dr = \frac{\pi}{2} \frac{r^{2-2s}}{2-2s}\Big|_{R_1}^{\infty}\end{aligned}$$

$$= \frac{\pi}{2} \frac{1}{2s-2} \frac{1}{R_1^{2s-2}} = \frac{\pi}{4} \frac{1}{s-1} \left(\frac{1}{R-\sqrt{2}} \right)^{2s-2}.$$

A similar argument shows that

$$\sum_{\substack{m \geq 1, n \leq -1 \\ m^2+n^2 > R^2}} \left(\frac{1}{m^2+n^2} \right)^s \leq \frac{\pi}{4} \frac{1}{s-1} \left(\frac{1}{R-\sqrt{2}} \right)^{2s-2}. \tag{5.26}$$

Combining these estimates, we obtain

$$\sum_{b \in I_1, |b|>R} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z)) \leq \exp\left(\frac{s}{\sqrt{R^2-R}} \right) \left(\frac{R}{R-1} \right)^s$$
$$\cdot \left[\frac{1}{2s-1} \left(\frac{1}{R-1} \right)^{2s-1} + \frac{\pi}{2} \frac{1}{s-1} \left(\frac{1}{R-\sqrt{2}} \right)^{2s-2} \right] v_s(0) := \delta_{s,R} v_s(0).$$

The estimate for the sum over I_2 follows by a similar but simpler argument, since only the inequality in (5.26) is needed. □

Remark 5.4. If $\mathcal{B} \subset I_1$ is an infinite set, $s > \tau(\mathcal{B})$ and v_s is the corresponding strictly positive eigenfunction of L_s in (3.1), an examination of the proof of Theorem 5.12 shows that

$$\sum_{b \in \mathcal{B}, |b|>R} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z))$$
$$\leq \exp\left(\frac{s}{\sqrt{R^2-R}} \right) \left(\frac{R}{R-1} \right)^s \left(\sum_{b \in \mathcal{B}, |b|>R} \frac{1}{|b|^{2s}} \right) v_s(0),$$

so an estimate for $\delta_{s,R}$ in this case will follow from an upper bound on $\sum_{\substack{b \in \mathcal{B} \\ |b|>R}} \frac{1}{|b|^{2s}}$.

It remains to estimate $\eta_{s,R}$ in Theorem 3.3. We could, of course, take $\eta_{s,R} = 0$, but we can do slightly better. Since the argument is similar to that in Theorem 5.12, we just sketch the proof.

Theorem 5.13. *Assume that $\mathcal{B}$ is an infinite subset of I_1, that $s > \tau(\mathcal{B})$, and that v_s is the strictly positive eigenfunction of L_s in* (3.1)*, where we take $U \supset D$ such that $0 \leq x \leq 1$ and $|y| \leq 1/2$ for all $(x, y) \in \bar{U}$. Then we have that*

$$\sum_{\substack{b \in \mathcal{B} \\ |b|>R}} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z))$$
$$\geq \exp\left(\frac{-\sqrt{5}s}{\sqrt{R^2-R}} \right) \left(\frac{R}{R+\sqrt{5}+[5/(4R)]} \right)^s v_s(0) \sum_{\substack{b \in \mathcal{B} \\ |b|>R}} \frac{1}{|b|^{2s}}$$
$$:= C(R,s) v_s(0) \sum_{b \in \mathcal{B}, |b|>R} \frac{1}{|b|^{2s}}.$$

If $\mathcal{B} = I_1$, $s > 1$ *and* $\theta_R = \arcsin(1/(R+\sqrt{2}))$,

$$\sum_{\substack{b\in I_1\\ |b|>R}} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z)) \geq C(R,s) v_s(0)(\pi - 2\theta_R)\Big(\frac{1}{2s-2}\Big)\Big(\frac{1}{R+\sqrt{2}}\Big)^{2s-2} := \eta_{s,R} v_s(0).$$

If $\mathcal{B} = I_2$ *and* $s > 1$,

$$\sum_{\substack{b\in I_2\\ |b|>R}} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z)) \geq C(R,s) v_s(0)(\pi/2 - 2\theta_R)\Big(\frac{1}{2s-2}\Big)\Big(\frac{1}{R+\sqrt{2}}\Big)^{2s-2} := \eta_{s,R} v_s(0).$$

Proof. By using (3.3) and the estimate in the proof of Theorem 5.12 that $1/|z+b|^2 \leq 1/(R^2 - R)$ for $|b| \geq R$ and $z \in \bar{U}$, we get

$$\sum_{\substack{b\in \mathcal{B}\\ |b|>R}} \frac{1}{|z+b|^{2s}} v_s(\theta_b(z)) \geq \exp\Big(\frac{-\sqrt{5}s}{\sqrt{R^2-R}}\Big) v_s(0) \sum_{b\in\mathcal{B}} \frac{1}{|z+b|^{2s}}.$$

If $b \in \mathcal{B}$, $|b| > R$, and $z \in \bar{U}$, one can check that

$$|z+b|^2 \leq \big[|b|^2(4R^2 + 4\sqrt{5}R + 5)\big]/[4R^2],$$

and this gives the first inequality in Theorem 5.13. If $b = m + n\mathrm{i} \in I_1$, let $\hat{b} = (m+1) + (n+1)\mathrm{i}$ if $n \geq 0$ and $\hat{b} = (m+1) + (n-1)\mathrm{i}$ if $n < 0$. Let $G_R = \{(x,y) \in \mathbb{R}^2 : x > 1 \text{ and } \sqrt{x^2+y^2} \geq R + \sqrt{2}\}$. One can check that

$$\sum_{\substack{b\in I_1\\ |b|>R}} \frac{1}{|b|^{2s}} \geq \sum_{\substack{b\in I_1\\ |\hat{b}|>R+\sqrt{2}}} \frac{1}{|b|^{2s}} \geq \int_{G_R} \Big(\frac{1}{x^2+y^2}\Big)^s dx\, dy,$$

and using polar coordinates gives the second inequality in Theorem 5.13. For I_2, let $H_R = \{(x,y) \in \mathbb{R}^2 : x > 1, y < -1, \text{ and } \sqrt{x^2+y^2} > R + \sqrt{2}\}$. One can check that

$$\sum_{\substack{b\in I_2\\ |b|>R}} \frac{1}{|b|^{2s}} \geq \sum_{\substack{b\in I_2\\ |\hat{b}|>R+\sqrt{2}}} \frac{1}{|b|^{2s}} \geq \int_{H_R} \Big(\frac{1}{x^2+y^2}\Big)^s dx\, dy,$$

and one obtains the final inequality in Theorem 5.13 with the aid of polar coordinates. □

Once the mesh size h has been chosen and $R > 2$ has been chosen (if $\mathcal{B} \subset I_1$ is infinite), the above results give formulas for nonnegative square matrices A_s and B_s such that $r(A_s) \leq r(L_s) \leq r(B_s)$, where L_s is as in (3.1). In particular, for $\mathcal{B} = I_1$, I_2, or I_3, if $r(A_{s_2}) > 1$ and $r(A_{s_2})$ is very close to one and $r(B_{s_1}) < 1$ and $r(B_{s_1})$ is very close to one, then the Hausdorff dimension s_* of the invariant set corresponding to $\mathcal{B}$ satisfies $s_2 < s_* < s_1$. Here s_2 and s_1 are obtained as described earlier.

Remark 5.5. For the set I_1 and $s = 1.86$, evaluating the above expressions gives for $\delta_{s,R}$ and $\eta_{s,R}$ the values

$$R = 100 : \delta_{s,R} = .00071,\ R = 200 : \delta_{s,R} = .00021,\ R = 300 : \delta_{s,R} = .00010,$$
$$R = 100 : \eta_{s,R} = .00059,\ R = 200 : \eta_{s,R} = .00019,\ R = 300 : \eta_{s,R} = .000096.$$

For the set I_2 and $s = 1.49$, evaluating the above expressions gives for $\delta_{s,R}$ and $\eta_{s,R}$ the values

$$R = 100 : \delta_{s,R} = .0184,\ R = 200 : \delta_{s,R} = .0091,\ R = 300 : \delta_{s,R} = .0061,$$
$$R = 100 : \eta_{s,R} = .0160,\ R = 200 : \eta_{s,R} = .0085,\ R = 300 : \eta_{s,R} = .0058.$$

6. Computing the Spectral Radius of A_s and B_s

In previous sections, we have constructed matrices A_s and B_s such that $r(A_s) \le r(L_s) \le r(B_s)$. The $m \times m$ matrices A_s and B_s have nonnegative entries, so the Perron-Frobenius theory for such matrices implies that $r(B_s)$ is an eigenvalue of B_s with corresponding nonnegative eigenvector, with a similar statement for A_s. One might also hope that standard theory (see [39]) would imply that $r(B_s)$, respectively $r(A_s)$, is an eigenvalue of B_s with algebraic multiplicity one and that all other eigenvalues z of B_s (respectively, of A_s) satisfy $|z| < r(B_s)$ (respectively, $|z| < r(A_s)$). Indeed, this would be true if B_s were *primitive*, i.e., if B_s^k had all positive entries for some integer k. However, typically B_s has many zero columns and B_s is neither primitive nor *irreducible* (see [39]); and the same problem occurs for A_s. Nevertheless, the desirable spectral properties mentioned above are satisfied for both A_s and B_s. Furthermore B_s has an eigenvector w_s with all positive entries and with eigenvalue $r(B_s)$; and if x is any $m \times 1$ vector with all positive entries,

$$\lim_{k\to\infty} \frac{B_s^k(x)}{\|B_s^k(x)\|} = \frac{w_s}{\|w_s\|},$$

where the convergence rate is geometric. Of course, corresponding results hold for A_s. Such results justify standard numerical algorithms for approximating $r(B_s)$ and $r(A_s)$.

These results were proved in the one dimensional case in [13]. Similar theorems can be proved in the two dimensional case, but because the proofs are similar, we omit the argument in the two dimensional case. The basic point, however, is simple: Although A_s and B_s both map the cone K of nonnegative vectors in $\mathbb{R}^m$ into itself, K is *not* the natural cone in which such matrices should be studied. Instead, one proceeds by defining, for large positive real M, a cone $K_M \subset K$ such that $A_s(K_M) \subset K_M$ and $B_s(K_M) \subset K_M$. The cone K_M is the discrete analogue of a cone which has been used before in the infinite dimensional case (see [44], Section 5 of [40], Section 2 of [33] and [5]). Once one shows that $A_s(K_M) \subset K_M$ and $B_s(K_M) \subset K_M$, the desired spectral properties of A_s and B_s follow easily by the arguments used in the papers cited above. In a later paper, we shall consider higher order piecewise polynomial approximations to the positive eigenfunction v_s of L_s. We hope to show that although the corresponding matrices A_s and B_s

no longer have all nonnegative entries, it is still possible to obtain rigorous upper and lower bounds on the Hausdorff dimension.

7. Log Convexity of the Spectral Radius of Λ_s

For $s \in \mathbb{R}$, we define $\Lambda_s : X \to X := C^m(\bar{H})$ and $L_s : Y \to Y := C(\bar{H})$ by

$$(\Lambda_s(f))(x) = \sum_{\beta \in \mathcal{B}} (g_\beta(x))^s f(\theta_\beta(x)) \tag{7.1}$$

and

$$(L_s(f))(x) = \sum_{\beta \in \mathcal{B}} (g_\beta(x))^s f(\theta_\beta(x)). \tag{7.2}$$

In general, if V is a convex subset of a vector space X, we shall call a map $f : V \to [0, \infty)$ log convex if (i) $f(x) = 0$ for all $x \in V$ or (ii) $f(x) > 0$ for all $x \in V$ and $x \mapsto \log(f(x))$ is convex. Products of log convex functions are log convex, and Hölders inequality implies that sums of log convex functions are log convex.

The following result plays an important role in our numerical approximation scheme.

Theorem 7.1. *Assume that hypotheses (H4.1), (H4.2), and (H4.3) are satisfied with $m \geq 1$ and that $H \subset \mathbb{R}^n$ is a bounded, open mildly regular set. For $s \in \mathbb{R}$, let Λ_s and L_s be defined by* (7.1) *and* (7.2)*. Then we have that $s \mapsto r(\Lambda_s)$ is log convex, i.e., $s \mapsto log(r(\Lambda_s))$ is convex on* $[0, \infty)$.

A proof of this and related results can be found in many papers ([8,13–15,27,29,37,42]). Note that the terminology *super convexity* is used to denote log convexity in [27] and [29], presumably because any log convex function is convex, but not conversely. Theorem 7.1, while adequate for our immediate purposes, can be greatly generalized by a different argument that does not require existence of strictly positive eigenvectors. This generalization (which we omit) contains Kingman's matrix log convexity result in [29] as a special case.

In our applications, the map $s \mapsto r(L_s)$ will usually be strictly decreasing on an interval $[s_1, s_2]$ with $r(L_{s_1}) > 1$ and $r(L_{s_2}) < 1$, and we wish to find the unique $s_* \in (s_1, s_2)$ such that $r(L_{s_*}) = 1$. The following hypothesis insures that $s \mapsto r(L_s)$ is strictly decreasing for all S.
(H7.1): Assume that $g_\beta(\cdot)$, $\beta \in \mathcal{B}$ satisfy the conditions of (H4.1). Assume also that there exists an integer $\mu \geq 1$ such that $g_\omega(x) < 1$ for all $\omega \in \mathcal{B}_\mu$ and all $x \in \bar{H}$.

Theorem 7.2. *Assume hypotheses (H4.1), (H4.2), (H4.3), and (H7.1) and let H be mildly regular. Then the map $s \mapsto r(\Lambda_s)$, $s \in \mathbb{R}$, is strictly decreasing and real analytic and* $\lim_{s \to \infty} r(\Lambda_s) = 0$.

This proof of this result is given in [13,37], and elsewhere.

Remark 7.1. Assume that the assumptions of Theorem 7.2 are satisfied and define $\psi(x) = \log(r(L_s)) = \log(r(\Lambda_s))$ (where log denotes the natural logarithm), so $s \mapsto \psi(s)$ is a convex, strictly decreasing function with $\psi(0) > 1$ (unless $|\mathcal{B}| = p = 1$) and $\lim_{s\to\infty} \psi(s) = -\infty$. We are interested in finding the unique value of s such that $\psi(s) = 0$. In general suppose that $\psi : [s_1, s_2] \to \mathbb{R}$ is a continuous, strictly decreasing, convex function such that $\psi(s_1) > 0$ and $\psi(s_2) < 0$, so there exists a unique $s = s_* \in (s_1, s_2)$ with $\psi(s_*) = 0$. If t_1 and t_2 are chosen so that $s_1 \le t_1 < t_2 \le s_*$ and t_{k+1} is obtained from t_{k-1} and t_k by the secant method, an elementary argument show that $\lim_{k\to\infty} t_k = s_*$. If $s_* \le t_2 < t_1 < s_2$ and $s_1 \le t_3$, a similar argument shows that $\lim_{k\to\infty} t_k = s_*$. If $\psi \in C^3$, elementary numerical analysis implies that the rate of convergence is faster than linear $(= (1+\sqrt{5})/2)$. In our numerical work, we apply these observations, not directly to $\psi(s) = \log(r(\Lambda_s))$, but to decreasing functions which closely approximate $\log(r(\Lambda_s))$.

One can also ask whether the maps $s \mapsto r(B_s)$ and $s \mapsto r(A_s)$ are log convex, where A_s and B_s are the previously described approximating matrices for L_s. An easier question is whether the map $s \mapsto r(M_s)$ is log convex, where A_s and B_s are obtained from M_s by adding error correction terms. In [13], it was proved that in the one dimensional case, $s \mapsto r(M_s)$ is log convex. The proof in the two dimensional case is similar, and we do not repeat it here.

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